Publ. Math. Debrecen
43 / 3–4 (1993), 273–282

On the generalized Einstein – Yang Mills equations

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Abstract. Let $\xi = (E, p, M)$ be a vector bundle, with basis M — a real differentiable manifold of dimension n, and fiber F of dimension m. Considering the automorphisms of ξ as gauge transformations, and the set of gauge fields $\{N_i^a(x,y)$, $L^i_{jk}(x,y)$, $L^a_{bk}(x,y)$, $C^i_{ja}(x,y)$, $C^a_{bc}(x,y)$, $g_{ij}(x,y)$, $h_{ab}(x,y)\}$ given by a nonlinear connection, a gauge linear d-connection [9,11], and a pair of metric gauge tensor fields in local adapted coordinates, the author obtains the form of the generalized Einstein–Yang Mills equations for the general case and for the quasi-metric h- and v-symmetrical cases. These results generalise the ones obtained by G.S. Asanov in [2,3], in a natural manner, basically using the formalism, notations and mathematical theory of distinguished geometrical object fields introduced by R. Miron [10, 11].

Let $\{N_i^a(x,y)\}$ be the coefficients of a nonlinear connection on the vector bundle $\xi = (E, p, M)$ in local coordinates (x^i, y^a), $i = \overline{1,n}$, $a = \overline{1,m}$ [8,11].

Definition 1. A local adapted basis in $\mathfrak{X}(E)$ is the set of vector fields $\{\delta_i, \dot\partial_a\}$, $i = \overline{1,n}$, $a = \overline{1,m}$, where

$$\delta_i = \frac{\partial}{\partial x^i} - N_i^a \frac{\partial}{\partial y^a}, \qquad \dot\partial_a = \frac{\partial}{\partial y^a}. \tag{1}$$

Definition 2. A linear d-connection on E is a linear connection ∇ that preserves the horizontal and the vertical distributions locally generated by $\{\delta_i, i = \overline{1,n}\}$ and $\{\dot\partial_a, a = \overline{1,m}\}$ respectively; in the local adapted basis (1) its coefficients are given by

$$\{L^i_{jk}(x,y),\ L^a_{bk}(x,y),\ C^i_{ja}(x,y),\ C^a_{bc}(x,y)\}$$

1980 *Mathematics Subject Classification* (1985 *Revision*): 53C60.
Keywords: gauge fields, gauge covariant derivations, generalized Einstein–Yang Mills equations.
This paper was presented at the Conference on Finsler Geometry and its Applications to Physics and Control Theory, August 26–31, 1991, Debrecen, Hungary.

where

$$\tag{2} \begin{cases} \nabla_{\delta_j}\delta_i = L^k_{ij}\delta_k, & \nabla_{\delta_j}\dot{\partial}_a = L^b_{aj}\dot{\partial}_b \\ \nabla_{\dot{\partial}_a}\delta_j = C^i_{ja}\delta_i, & \nabla_{\dot{\partial}_b}\dot{\partial}_a = C^d_{ab}\dot{\partial}_d \end{cases}$$

Definition 3. The h- and v-covariant derivation laws associated to the linear d-connection (2) are defined by

$$\tag{3} \begin{cases} D_i w^{ma}_{nb} = \delta_i w^{ma}_{nb} + L^m_{ki} w^{ka}_{nb} - L^k_{ni} w^{ma}_{kb} + L^a_{di} w^{md}_{nb} - L^d_{bi} w^{ma}_{nd} \\ D_c w^{ma}_{nb} = \dot{\partial}_c w^{ma}_{nb} + C^m_{kc} w^{ka}_{nb} - C^k_{nc} w^{ma}_{kb} + C^a_{dc} w^{md}_{nb} - C^d_{bc} w^{ma}_{nd}. \end{cases}$$

Proposition 1. *The transformation rules for the coefficients of the linear d-connection are*

$$\tag{4} \begin{aligned} &\bar{\partial}_m B^k_j - B^k_i \bar{L}^i_{jm}(\bar{x},\bar{y}) + B^i_j B^n_m L^k_{in}(x,y) = 0 \\ &\bar{\partial}_m M^a_b - M^a_c \bar{L}^c_{bm}(\bar{x},\bar{y}) + M^c_b B^n_m L^a_{cn}(x,y) = 0 \\ &B^i_n \bar{C}^n_{ma}(\bar{x},\bar{y}) = M^c_a B^j_m C^i_{jc}(x,y) \\ &M^a_d \bar{C}^d_{bc}(\bar{x},\bar{y}) = M^d_b M^f_c C^a_{df}(x,y) \end{aligned}$$

where the coordinate transformations on E have the form

$$\tag{5} \begin{aligned} x^i &= x^i(\bar{x}), \det(\partial x^i/\partial \bar{x}^j) \neq 0 \\ y^a &= M^a_b(\bar{x})\bar{y}^b, \det(M^a_b(\bar{x})) \neq 0 \end{aligned}$$

and we used the notations

$$B^i_j = \bar{\partial}_j x^i, \quad \bar{\partial}_j = \frac{\partial}{\partial \bar{x}^j}$$

Definition 4. A gauge transformation is a automorphism of E [7,2,3], *locally given by*

$$\tag{6} \begin{aligned} x^i &= X^i(\tilde{x}), \det(\tilde{\partial}_j X^i) \neq 0 \\ y^a &= Y^a(\tilde{x},\tilde{y}), \det(Y^a_b) \neq 0, \quad \dot{\tilde{\partial}}_c Y^a_b = 0 \end{aligned}$$

where we denoted

$$Y^a_b = \dot{\tilde{\partial}}_b Y^a, \quad \dot{\tilde{\partial}}_b = \frac{\partial}{\partial \tilde{y}^b}, \quad \tilde{\partial}_j = \frac{\partial}{\partial \tilde{x}^j}$$

Definition 5. A (generalized [2,3]) gauge tensor field is a field on E, which obeys tensorial rules of transformation relative to (5) and (6); e.g. $\left\{w^{ia}_{jb}\right\}$ obeys

$$\tag{7} \begin{aligned} B^i_k M^a_c \bar{w}^{kc}_{jb} &= B^\ell_j M^d_b w^{ia}_{\ell d} \\ X^i_k Y^a_c \tilde{w}^{kc}_{jb} &= X^\ell_j Y^d_b w^{ia}_{\ell d}, \quad \text{where} \quad X^i_k = \tilde{\partial}_k X^i. \end{aligned}$$

Definition 6. A gauge covariant derivation (h-resp. v-) is given by the h- and v-derivation laws in definition 3, which preserves the gauge tensorial character relative to (5), (6).

Proposition 2. *The coefficients of the h- and v-gauge covariant derivations have with respect to* (6) *the transformation laws*

$$\begin{aligned}
&\tilde{\partial}_m X^k_j - X^k_i \tilde{L}^i_{jm}(\tilde{x},\tilde{y}) + X^i_j X^n_m L^k_{in}(x,y) = 0\\
&\tilde{\partial}_m Y^a_b - Y^a_c \tilde{L}^c_{bm}(\tilde{x},\tilde{y}) + Y^c_b X^n_m L^a_{cn}(x,y) = 0\\
&X^i_n \tilde{C}^n_{ka}(\tilde{x},\tilde{y}) = Y^c_a X^j_k C^i_{jc}(x,y)\\
&Y^a_d \tilde{C}^d_{bc}(\tilde{x},\tilde{y}) = Y^d_b Y^f_c C^a_{df}(x,y)
\end{aligned} \tag{8}$$

Remarks.
1. $\{C^i_{ja}\}$ and $\{C^a_{bc}\}$ are gauge tensor fields.
2. The coefficients $\{L^i_{jk}, L^a_{bk}, C^i_{ja}, C^a_{bc}\}$ of the h- and v-gauge covariant derivations (3) are in fact the coefficients of a linear d-connection which satisfies the supplementary rules (8) (gauge linear d-connection).

Proposition 3. *The torsion and the curvature gauge tensor fields of a gauge linear d-connection are given by* [11]

$$\begin{gathered}
T^i_{jk} = L^i_{[jk]}, \quad R^a_{jk} = -\delta_{[j} N^a_{k]}, \quad P^i_{jc} = C^i_{jc}\\
P^a_{jb} = \dot{\partial}_b N^a_j - L^a_{bj}, \quad S^a_{bc} = C^a_{[bc]}
\end{gathered} \tag{9}$$

and respectively

$$\begin{aligned}
R^i_{jk\ell} &= \delta_{[\ell} L^i_{jk]} + L^h_{j[k} L^i_{h\ell]} + C^i_{ja} R^a_{k\ell},\\
R^a_{bk\ell} &= \delta_{[\ell} L^a_{bk]} + L^c_{b[k} L^a_{c\ell]} + C^a_{bc} R^c_{k\ell},\\
P^i_{jkc} &= \dot{\partial}_c L^i_{jk} - D_k C^i_{jc} + C^i_{jb} P^b_{kc},\\
P^a_{bkc} &= \dot{\partial}_c L^a_{bk} - D_k C^a_{bc} + C^a_{bd} P^d_{kc},\\
S^i_{jbc} &= \dot{\partial}_{[c} C^i_{jb]} + C^h_{j[b} C^i_{hc]}\\
S^a_{bcd} &= \dot{\partial}_{[c} C^a_{bd]} + C^e_{b[c} C^a_{ed]}
\end{aligned} \tag{9'}$$

where we used the notation for $[i \dots j]$*:*

$$L^h_{k[i} L^s_{hj]} = L^h_{ki} L^s_{hj} - L^h_{kj} L^s_{hi}$$

Proposition 4. *The following mixed Lagrangian is invariant under* (5) *and* (6) *(i.e. it is a scalar gauge field)*

$$L = \sum_{i \in I} n_i \cdot L_i, \quad n_i \in \mathbb{R}, \qquad i \in I = \{\overline{1,5}, \overline{11,16}, \overline{21,22}\} \tag{10}$$

where

$$
\begin{aligned}
&L_1 = T^i_{jk}T^{jk}_i, \quad L_2 = R^a_{jk}R^{jk}_a, \quad L_3 = P^i_{jc}P^{jc}_i,\ L_4 = P^a_{jb}P^{jb}_a\\
&L_5 = S^a_{bc}S^{bc}_a, \quad L_{21} = R^i_{jkl}R^{jkl}_i,\ L_{22} = S^a_{bcd}S^{bcd}_a,\\
&L_{11} = R^{ij}{}_{ij}, \quad L_{12} = R^a_{bk\ell}R^{bk\ell}_a,\ L_{13} = P^i_{jkc}P^{jkc}_i\\
&L_{14} = P^a_{bkc}P^{bkc}_a,\ L_{15} = S^i_{jbc}S^{jbc}_i,\ L_{16} = S^{ab}{}_{ab}
\end{aligned} \tag{11}
$$

The proof is computational.

Remark. The Lagrangian L contains, relative to [2], the supplementary terms n_3L_3 and $n_{15}L_{15}$, and the terms $n_{11}L_{11}$ and $n_{13}L_{13}$ are altered (are more general) since the present context doesn't impose the restrictive condition $C^i_{ja} = 0$.

The raising/lowering of the corresponding indices in (11) were performed via the gauge metric tensor fields $\{g_{ij}(x,y)\}$ and $\{h_{ab}(x,y)\}$ [3,2]. Then, introducing the Lagrangian density

$$\mathcal{L} = LG \tag{12}$$

where $G = |\det(g_{ij})|^{1/2} \cdot |\det(h_{ab})|^{1/2}$, we notice that it depends on the gauge fields

$$\phi \in \{N^a_i,\ L^i_{jk},\ L^a_{bk},\ C^i_{ja},\ C^a_{bc},\ g_{ij},\ h_{ab}\} \tag{13}$$

and their derivatives; considering the variational principle [1,3]

$$\delta \int \mathcal{L} dx^n dy^m = 0$$

one can derive the extremum condition of vanishing the Euler–Lagrange derivatives

$$\frac{\delta\mathcal{L}}{\delta\phi} \equiv \frac{\partial}{\partial x^j}\left(\frac{\partial\mathcal{L}}{\partial(\partial_j\phi)}\right) + \frac{\partial}{\partial y^a}\left(\frac{\partial\mathcal{L}}{\partial(\dot\partial_a\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0.$$

Theorem 1. *The generalized Einstein–Yang Mills equations associated to the Lagrangian* (10) *for the set of arguments* (13) *are*

$$
\begin{aligned}
\frac{\delta L}{\delta N^a_k} &= -4n_2(D^*_\ell R^{\ell k}_a + P^c_{a\ell}R^{\ell k}_c + \frac{1}{2}T^k_{n\ell}R^{n\ell}_a) -\\
&\quad -2n_4(D^*_b P^{bk}_a + C^c_{ab}P^{bk}_c + \underline{P^k_{nb}P^{nb}_a}) -\\
&-4\underline{n_{21}}[D^*_\ell V^{\ell k}_a + P^c_{a\ell}V^{\ell k}_c + \frac{1}{2}T^k_{n\ell}V^{n\ell}_a - R^{j\ell k}_i(P^i_{j\ell a} + D_\ell P^i_{ja} - P^i_{jb}P^b_{\ell a})] -\\
&-n_{11}[D^*_\ell U^{\ell k}_a + P^c_{a\ell}U^{\ell k}_c + \frac{1}{2}T^k_{n\ell}U^{n\ell}_a - w^{j[lk]}_i(P^i_{j\ell a} + \underline{D_\ell P^i_{ja} - P^i_{jb}P^b_{\ell a}})] +
\end{aligned} \tag{15.1}
$$

$$+4n_{12}[P^c_{b\ell a}R^{b\ell k}_c + C^c_{ba}\cdot\frac{1}{4}\delta L_{12}/\delta L^c_{bk}]+$$
$$+2\underline{n_{13}}[S^i_{jac}P^{jkc}_i + (D^*_c P^{jkc}_i - P^k_{nc}P^{jnc}_i)P^i_{ja}]+$$
$$+2n_{14}[S^d_{bac}P^{bkc}_d + (D^*_c P^{bkc}_d - P^k_{nc}P^{bnc}_d)C^d_{ba}] = 0,$$

$$\frac{\delta L}{\delta L^i_{jk}} = -4n_1T^{jk}_i + 4\underline{n_{21}}(D^*_\ell R^{jk\ell}_i - \frac{1}{2}T^k_{n\ell}R^{jn\ell}_i) \tag{15.2}$$
$$+n_{11}(D^*_\ell w^{j[k\ell]}_i - \frac{1}{2}T^k_{n\ell}w^{j[n\ell]}_i) + 2\underline{n_{13}}(D^*_c P^{jkc}_i - P^k_{nc}P^{jnc}_i) = 0,$$

$$\begin{aligned}\frac{\delta L}{\delta L^a_{bk}} =& -2n_4P^{bk}_a + 4n_{12}(D^*_\ell R^{bk\ell}_a - \frac{1}{2}T^k_{n\ell}R^{bn\ell}_a)\\ &+2n_{14}(D^*_c P^{bkc}_a - \underline{P^k_{nc}P^{bnc}_a}) = 0,\end{aligned} \tag{15.3}$$

$$\begin{aligned}\frac{\delta L}{\delta C^i_{ja}} =& -2\underline{n_3}P^{ja}_i + 2\underline{n_{21}}R^a_{k\ell}R^{jk\ell}_i - n_{11}R^{ja}{}_i -\\ &-2n_{13}(D^*_\ell P^{j\ell a}_i - P^a_{b\ell}P^{j\ell b}_i)-\\ &-4\underline{n_{15}}(D^*_d S^{jda}_i - \frac{1}{2}S^a_{bd}S^{jdb}_i) = 0,\end{aligned} \tag{15.4}$$

$$\frac{\delta L}{\delta C^a_{bc}} = -4n_5S^{bc}_a + 2n_{12}R^c_{k\ell}R^{bk\ell}_a + \underline{4n_{22}}(D^*_d S^{bcd}_a - \frac{1}{2}S^c_{ed}S^{bed}_a)+ \tag{15.5}$$
$$+n_{16}(D^*_d w^{b[cd]}_a - \frac{1}{2}S^c_{ed}w^{b[ed]}_a) - 2n_{14}(D^*_\ell P^{b\ell c}_a - P^c_{d\ell}P^{b\ell d}_a) = 0,$$

$$\frac{\delta L}{\delta g_{ij}} = -\frac{\partial L}{\partial g_{ij}} - \frac{1}{2}g^{ij}L = 0, \tag{15.6}$$

$$\frac{\delta L}{\delta h_{ab}} = -\frac{\partial L}{\partial h_{ab}} - \frac{1}{2}h^{ab}L = 0, \tag{15.7 }$$

where we denoted

$$\begin{cases} D^*_\ell = D_\ell + V_\ell, \quad V_\ell = \dfrac{D_\ell G}{G} + T^n_{n\ell} + P^d_{d\ell}\\ D^*_c = D_c + V_c, \quad V_c = \dfrac{D_c G}{G} + P^n_{nc} + S^d_{dc}\end{cases}$$

$$V^{k\ell}_a = C^i_{ja}R^{jk\ell}_i,\ U^{k\ell}_a = C^i_{ja}w^{j[k\ell]}_i$$
$$w^{jk\ell}_i = g^{jk}\delta^\ell_i,\ w^{bcd}_a = h^{bc}\delta^d_a; \qquad \frac{\delta L}{\delta\phi} \equiv \frac{1}{G}\frac{\delta\mathcal{L}}{\delta\phi}.$$

Hint. The Euler–Lagrange derivatives for the elementary Lagrangians (11) computed using the relations

$$D_k G = \delta_k G - G(L^n_{nk} + L^a_{ak})$$
$$D_a G = \dot{\partial}_a G - G(C^n_{na} + C^d_{da}).$$

give by addition the equations above.

Remark. The underscored terms are new with respect to [2], and the equations in [1,2] can be viewed as a particular case of (15.1)–(15.7). The notations of tensor fields and vertical indices are changed from those used by G.S. Asanov to the corresponding ones used in the papers [8,9,11] in the theory of Finsler spaces. Also, the fact that in [1,2] the coefficients of the nonlinear connection considered in (1) are taken with opposite sign, induce related differences of sign in (15.1)–(15.7).

In the following we consider the quasi-metric case, i.e. the situation in which the gauge metric tensor fields obey

$$D_k g_{ij} = 0, \quad D_k h_{ab} = 0, \quad D_c h_{ab} = 0, \tag{16}$$

and impose for the gauge linear d-connection (2) to be h- and v-symmetrical, i.e. $T^i_{jk} = 0$ and $s^a_{bc} = 0$. Then we can state the following

Theorem 2. *The generalized Einstein–Yang Mills equations in the quasi-metric h- and v-symmetrical case, for the generalized gauge Lagrangian* [2]

$$\begin{aligned} L = & n_1 R^a_{jk} R^{jk}_a + n_3 P^a_{jb} P^{jb}_a + \ell_1 R^{mn}{}_{mn} + \ell_2 R^{abk\ell} R_{abk\ell} + \\ & + \ell_{10} S^{ab}{}_{ab} + \Lambda, \qquad \Lambda \in \mathcal{F}(E) \end{aligned} \tag{17}$$

with respect to the set of arguments

$$\phi \in \{N^a_i, A_{abi} = \frac{1}{2} L_{[ab]i}, C^i_{ja}, g_{ij}, h_{ab}\} \tag{18}$$

are the following

$$\begin{aligned} \frac{\delta L}{\delta N^a_i} = & 4n_1(D^*_m R^{im}_a - P^b_{ka} R^{ik}_b) + 2n_3(D^*_b P^{ib}_a - \underline{P^{nb}_a P^i_{nb}}) + \\ & + \ell_1[\underline{D^*_j U^{ij}_a + U^{ij}_b P^b_{aj}} - w^{nji}_k(-\dot{\partial}_a L^k_{nj} + V_n g^{km} \dot{\partial}_a g_{mj})] + \\ & + 4\ell_2 R^{bcki} P_{bcka} + \frac{1}{2} C_{cba} \cdot \frac{\delta L}{\delta A_{cbi}} = 0, \end{aligned} \tag{19.1}$$

$$\begin{aligned} \frac{\delta L}{\delta g_{ij}} = & -\frac{1}{2} g^{ij} L + 2n_1 R^{iak} R^j_{ak} + n_3 P^{iab} P^j_{ab} + 2\ell_2 R_{abk}{}^i R^{abkj} + \\ & + \ell_1(R^{inj}{}_n + g^{ij} D^*_k V^k - \frac{1}{2} D^{*\{i} V^{j\}} + \frac{1}{2} \underline{P^{\{in}{}_a R^{j\}a}{}_n}) = 0, \end{aligned} \tag{19.2}$$

$$
\begin{aligned}
\frac{\delta L}{\delta h_{ab}} = & -\frac{1}{2} h^{ab} L - n_1 R^{ka\ell} R^b_{k\ell} + \ell_2 D^*_c R^{\{b}_{k\ell} R^{ca\}k\ell} - \frac{1}{2} L^{\{a}_{dj} \frac{\delta L}{\delta A_{db\}j}} + \\
& + n_3 (P^a_{jc} P^{jcb} - P^a_{jc} P^{jbc} - \frac{1}{2} D^*_j P^{j\{ab\}}) + \\
& + \ell_{10} (S^{acb}{}_c + h^{ab} D^*_c V^c - \frac{1}{2} D^{*\{a} V^{b\}}) = 0,
\end{aligned} \tag{19.3}
$$

$$
\frac{\delta L}{\delta A_{abi}} = n_3 P^{i[ba]} - 4\ell_2 D^*_n R^{abni} = 0, \tag{19.4}
$$

$$
\frac{\delta L}{\delta C^i_{ja}} = \ell_1 R_i{}^{aj} = 0, \tag{19.5}
$$

where we denoted

$$
\begin{cases}
D^*_\ell = D_\ell + V_\ell, & V_\ell = P^d_{d\ell} \\
D^*_a = D_a + V_a, & V_a = \dot{\partial}_a \ln \sqrt{|\det(g_{ij})|}
\end{cases}
$$

$$
\begin{aligned}
U^{jk}_a &= C_{nma} w^{nmjk}, \quad w^{nmjk} = g^{n[j} g^{mk]} \\
\delta L / \delta \phi &= \frac{1}{G} \frac{\delta \mathcal{L}}{\delta \phi}; \quad \tau_{\{ij\}} = \tau_{ij} + \tau_{ji}; \quad \tau_{[ij]} = \tau_{ij} - \tau_{ji}\,.
\end{aligned}
$$

Hint. The same procedure as in Theorem 1 can be applied, using the relations [2,11]:

$$
\begin{aligned}
L_{abi} &= h_{bs} L^s_{ai} = A_{abi} + \frac{1}{2} \delta_i h_{ab} \\
C^a_{bc} &= \frac{1}{2} h^{ad} (\dot{\partial}_{\{b} h_{dc\}} - \dot{\partial}_d h_{bc}).
\end{aligned}
$$

Remark. As in theorem 1, the vanishing of the underscored terms in (19.1)–(19.5) give, as a particular case, the corresponding equations in [2].

The attempt of solving the equations in theorem 2 leads to the following results

Theorem 3. *The generalized Einstein–Yang Mills equations* (19.1)–(19.5) *admit the solution*

$$
\{N^a_i, g_{ij}, h_{ab}, A_{abk}, C^i_{ja}\} \tag{20}
$$

given by

$$
(21)\qquad
\begin{aligned}
&N_i^a = -\frac{C_i y^a}{2C}, C \in \mathcal{F}(M), \quad D_k^*\left(\frac{C_k}{C}\right) \neq 0\\
&g_{ij} = e^{\lambda(v)} \cdot \tau_{ij}(x), \quad v \in \mathcal{F}(E)\\
&h_{ab} = \gamma_{ab}(x) + b(x,y)y_a y_b, \quad \textit{with} \;\; y_a \equiv \gamma_{ab} y^b, b(x,y) = \frac{1}{v^2}\\
&A_{abk} = 0\\
&C^i_{ja} \;\; \textit{satisfying:} \;\; D_j U_a^{ij} = \eta_j U_a^{ij}, \; U_a^{ij} \equiv C_{nma} g^{n[i} g^{mj]}
\end{aligned}
$$

where $\tau_{ij}(x)$ *satisfies the Einstein equations of Riemannian type*

$$
(22)\qquad E_{ij} = m[\tau_{ij}(\alpha p + \nabla_k \alpha^k) - q^{\alpha_i \alpha_j} - \frac{1}{2}\nabla_{\{i}\alpha_{j\}}]
$$

and (21) *are subject to the following conditions*

$$
(23)\qquad \delta_k \gamma_{ab} = 0, \quad v = [C(x) - y^2]^{-1/2}
$$

$$
\lambda(x,y) = \frac{2}{n}\ln\left(v\sqrt{\frac{k}{C(x)}}\right), \quad \textit{with} \;\; k \in \mathbb{R}^*_+, \; C(x) > y^2, \; (y^2 \equiv y_a y^a)
$$

$$
\eta \equiv d\left[\ln(|C(x)|^{(1-m)/2})\right], \; \Lambda = -m(m-1)\ell_{10}/C(x).
$$

In (21)–(23) ϱ_{ij}, ϱ *and* ∇_k *are the Ricci tensor field, the scalar curvature and the covariant derivative associated to* τ_{ij}, *and we used the notations*

$$
\begin{cases}
C_i = \dfrac{\partial C}{\partial x^i}, \quad \alpha_i = -\dfrac{C_i}{2C}, \quad \alpha = \alpha_i \alpha_j \gamma^{ij}\\
E_{ij} = \varrho_{ij} - \dfrac{1}{2}\varrho\tau_{ij} \;\; \textit{(the Einstein tensor field)}\\
p = \dfrac{m(3-n)}{2(n-2)}, \quad q = \dfrac{m}{n-2}
\end{cases}
$$

Proof. The equations (19.1)–(19.5) have a general form; it becomes possible to search for solutions of the family (22) of the form (23)–(23') under additional simplifying assumptions, namely

(A1) $\delta_k h_{ab} = 0$

(A2) $b = \dfrac{1}{v^2} = b(x,z)$, with $z = y^2$

(A3) the differential equation of Riccati type obtained from $\dfrac{\delta L}{\delta h_{ab}} = 0$, to become one of Bernoulli type

(A4) $\delta_k \lambda = 0$, $\delta_k \gamma_{ab} = 0$.

The cosmologycal constant Λ is obtained from $\dfrac{\delta L}{\delta g_{ij}} = 0$, equation which provides the classical Einstein equations in (24). The conditions (A1)–(A4) yield to the form of the class of solutions stated in the theorem.

For the case when ξ is the tangent bundle of M, and $L^n = (M, g_{ij}(x,y))$ is a structure of a generalised Lagrange space [11] endowed with the non-linear connection $\{N_i^a(x,y)\}$

$$N_i^a = \begin{Bmatrix} a \\ ij \end{Bmatrix} y^j, \ \begin{Bmatrix} i \\ jk \end{Bmatrix} = \frac{1}{2}\gamma^{is}\left(\frac{\partial \gamma_{sj}}{\partial x^k} + \frac{\partial \gamma_{sk}}{\partial x^j} - \frac{\partial \gamma_{jk}}{\partial x^s}\right) \tag{24}$$

and the fundamental tensor field

$$g_{ij}(x,y) = \gamma_{ij}(x) + \frac{1}{c^2} y_i y_j; \quad y_i = \gamma_{is} y^s, \quad c > 0, \tag{25}$$

where $\{\gamma_{ij}(x)\}$ is a Riemannian metric on M, we consider the N-lift of g_{ij} to TM ([11])

$$G = g_{ij}(x,y) dx^i \otimes dx^j + g_{ab}(x,y) \delta y^a \otimes \delta y^b. \tag{26}$$

Then, for the case of the canonic metrical h- and v-symmetric linear d-connection, we obtain the coefficients

$$\begin{aligned} L^i_{jk} &= \frac{1}{2} g^{in}(\delta_{\{j} g_{nk\}} - \delta_n g_{jk}) \\ C^a_{bc} &= \frac{1}{2} h^{ad}(\dot\partial_{\{b} h_{dc\}} - \dot\partial_d h_{bc}) \\ \text{with} \ \ h_{ab} &\equiv g_{ab} = g_{ij}\delta^i_a \delta^j_b, \ \ \text{and} \\ L^a_{bk} &= L^i_{jk}\delta^a_i \delta^j_b, \ C^i_{jc} = C^a_{bc}\delta^i_a \delta^b_j \end{aligned} \tag{27}$$

and can formulate the following

Theorem 4. *If L^n is locally Minkowskian, then the gauge fields* (13) *given by* (24), (25), (27) *provide solutions for the generalized Einstein–Yang Mills equations* (19.1)–(19.5) *iff $n = 2$ and*

$$\Lambda = 6\ell_{10}/(1 + 3y^2)$$

where $y^2 = \gamma_{ij} y^i y^j$ and $\left\{ {a \atop ij} \right\}$ are the Christoffel coefficients for $\gamma_{ij}(x)$, (see (24)*).*

Remark. The vanishing of the cosmological constant Λ would infer that (19.1)–(19.5) have no solution of the given form, unless $\ell_{10} = 0$.

Acknowledgement. The author is grateful to the referee for his remarks which improve the previous version of the paper.

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(Received January 22, 1992; revised August 10, 1992)