

On convexities of lattices

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Abstract. In this paper, some principal convexities of lattices and their mutual relations are investigated.

The notion of convexity of lattices has been introduced by E. Fried at the Problem session of the International Conference held in memory of Wilfried Nöbauer in Krems, Austria, in 1988 (cf. [9]). He also proposed a problem concerning the “number” of convexities of lattices. J. JAKUBÍK solved this problem in [4]; he showed that convexities of lattices form a proper class. In [4], it is also proved that the class of all convexities of lattices is a complete lattice (omitting the fact that it is a proper class) and the two-element chain generates an atom of this lattice.

Convexities can be defined also for various types of ordered algebraic structures. J. JAKUBÍK defined and studied convexities of d -groups [5] and l -groups ([6], [7]). Some results concerning convexities of Riesz groups were derived in [8].

In the present paper we investigate the relation between some principal convexities. We also touch the problem of atoms in the lattice of all convexities and we prove that this lattice is distributive. Finally we propose some open questions.

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1. Preliminaries

Let \mathcal{L} be the class of all lattices. A subclass \mathcal{K} of \mathcal{L} is said to be a convexity of lattices (or simply a convexity), whenever \mathcal{K} is closed under homomorphic images, convex sublattices and direct products. Comparing this notion with that of a variety of lattices, we see that each variety is a convexity. The converse does not hold in general. E.g., the convexity \mathcal{K} generated by a two-element chain is not a variety. Namely, in the opposite case, \mathcal{K} would have to contain all distributive lattices. But this is not true, because there exist infinitely many convexities of distributive lattices, \mathcal{K} being the least non-trivial one, as it follows from results of Section 3.

For a nonempty subclass \mathcal{X} of the class \mathcal{L} we denote by

$H\mathcal{X}$: the class of all homomorphic images of elements of \mathcal{X} ;

$C\mathcal{X}$: the class of all convex sublattices of elements of \mathcal{X} and their isomorphic copies;

$P\mathcal{X}$: the class of all direct products of elements of \mathcal{X} and their isomorphic copies.

We will use the following theorem (cf. [9] or [4]).

Theorem 1.1. *Let $\emptyset \neq \mathcal{X} \subseteq \mathcal{L}$. Then $HCP\mathcal{X}$ is a convexity; moreover, it is the least one containing \mathcal{X} .*

If \mathcal{X} is a one-element class, then the variety $HCP\mathcal{X}$ is said to be principal.

Let \mathcal{C} be the class of all convexities of lattices. It is partially ordered by the class-theoretical inclusion. It is easy to verify that if $\{\mathcal{K}_i : i \in I\}$ is a nonempty subclass of \mathcal{C} , then $\bigcap_{i \in I} \mathcal{K}_i$ is a convexity, too. In view of this and the fact that \mathcal{L} is the greatest element of \mathcal{C} , we will refer to \mathcal{C} as a complete lattice (omitting the fact that \mathcal{C} is a proper class). We will also apply the usual lattice-theoretical terminology and notation. So we will use the symbol $\bigvee_{i \in I} \mathcal{K}_i$ and $\bigwedge_{i \in I} \mathcal{K}_i$ for the least upper bound and the greatest lower bound of $\{\mathcal{K}_i : i \in I\} (\subseteq \mathcal{C})$, respectively. Evidently $\bigwedge_{i \in I} \mathcal{K}_i = \bigcap_{i \in I} \mathcal{K}_i$, $\bigvee_{i \in I} \mathcal{K}_i = HCP(\bigcup_{i \in I} \mathcal{K}_i)$.

As to the notation, \mathbb{N} will be the chain of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. \mathbb{Z} will be the chain of all integers, while the symbol \mathbb{R} will be used for the chain of all real numbers. The n -element chain ($n \in \mathbb{N}$) will be denoted by C_n .

2. Convexities generated by M_α

Let α be a cardinal, $\alpha \geq 3$. We denote by M_α the lattice consisting of elements $u, v, x_j (j \in J)$, where $\text{card } J = \alpha, u < x_j < v$ and $x_{j(1)}$ is incomparable with $x_{j(2)}$ whenever $j(1)$ and $j(2)$ are distinct elements of J . J. JAKUBÍK proved in [4] that if α, β are cardinals, $3 \leq \alpha < \beta$, then M_α does not belong to the convexity $HCP\{M_\beta\}$. We will show that if α, β are different finite cardinals, then $HCP\{M_\alpha\}$ and $HCP\{M_\beta\}$ are incomparable convexities, while for α, β infinite this is not the case in general.

If L_i is a lattice for each $i \in I, I \neq \emptyset$ and \mathcal{F} is a dual ideal of the lattice of all subsets of I , then the symbol $\prod(L_i \mid i \in I)/\mathcal{F}$ will be used for the reduced product of $(L_i \mid i \in I)$. If \mathcal{F} is an ultrafilter (also called prime dual ideal), then $\prod(L_i \mid i \in I)/\mathcal{F}$ will be also referred to as an ultraproduct of $(L_i \mid i \in I)$. The symbol $\theta(\mathcal{F})$ will be used for the congruence relation corresponding to \mathcal{F} , $[f]\mathcal{F}$ will mean the congruence class containing $f \in \prod(L_i \mid i \in I)$. (See e.g. [3]).

For a nonempty subclass \mathcal{X} of the class \mathcal{L} we denote by

$P_{\mathcal{U}}\mathcal{X}$: the class of all lattices that are isomorphic to an ultraproduct of members of \mathcal{X} .

The proof of the following theorem is a slight modification of that of the analogous theorem for varieties of lattices (cf. [3], p. 302, Theorem 9).

Theorem 2.1. *Let $\emptyset \neq \mathcal{X} \subseteq \mathcal{L}, \mathcal{K} = HCP\mathcal{X}$. If $L \in \mathcal{K}$ and L is subdirectly irreducible, then $L \in HCP_{\mathcal{U}}\mathcal{X}$.*

PROOF. Let L be a subdirectly irreducible lattice, $L \in \mathcal{K}$. Then there exist $L_i \in \mathcal{X}, i \in I$, a convex sublattice B of $\prod(L_i \mid i \in I)$, and a congruence relation ϕ on B such that L is isomorphic to B/ϕ . By the above mentioned result from [3], there exists an ultrafilter \mathcal{F} over I such that the corresponding congruence relation $\theta(\mathcal{F})$ restricted to B is contained in ϕ . Consider the set $\{[b]\mathcal{F} : b \in B\}$. It is evidently a sublattice of $\prod(L_i \mid i \in I)/\mathcal{F}$. We will show that it is convex. Let $[b_1]\mathcal{F} \leq [f]\mathcal{F} \leq [b_2]\mathcal{F}$ for some $b_1, b_2 \in B, f \in \prod(L_i \mid i \in I)$. We can suppose that $b_1 \leq b_2$ (in the opposite case we would take $b_1 \wedge b_2$ instead of b_1 and $b_1 \vee b_2$ instead of b_2). Let $g = (b_1 \vee f) \wedge b_2$. Then $b_1 \leq g \leq b_2$ yields $g \in B$. Hence $[f]\mathcal{F} = ([b_1]\mathcal{F} \vee [f]\mathcal{F}) \wedge [b_2]\mathcal{F} = [(b_1 \vee f) \wedge b_2]\mathcal{F} = [g]\mathcal{F}$, which belongs to $\{[b]\mathcal{F} : b \in B\}$. Now the correspondence $[b]\mathcal{F} \mapsto [b]\phi$ is a homomorphism of $\{[b]\mathcal{F} : b \in B\}$ onto B/ϕ . Thus $L \in HCP_{\mathcal{U}}\mathcal{X}$. \square

Corollary 2.2. *Let \mathcal{X} be a finite set of finite lattices. If $L \in HCP\mathcal{X}$ and L is subdirectly irreducible, then $L \in HC\mathcal{X}$.*

PROOF. Under our assumptions concerning \mathcal{X} , $P_{\mathcal{U}}\mathcal{X}$ is, up to isomorphic copies, \mathcal{X} . \square

Applying this theorem to $\mathcal{X} = \{M_\alpha\}$ for any finite cardinal α , $\alpha \geq 3$, we obtain that M_α and the two-element chain are the only subdirectly irreducible members of $HCP\{M_\alpha\}$. This implies

Corollary 2.3. *If α, β are any distinct finite cardinals, $\alpha, \beta \geq 3$, then the convexities $HCP\{M_\alpha\}$, $HCP\{M_\beta\}$ are incomparable.*

Further we will consider α to be an infinite cardinal number.

Lemma 2.4. *If $L \in P_{\mathcal{U}}\{M_\alpha\}$, then L is isomorphic to M_β for some $\beta \geq \alpha$.*

PROOF. Let $L = M_\alpha^I/\mathcal{F}$ for a nonempty set I and an ultrafilter \mathcal{F} over I . Since “being one of the M_γ for some $\gamma \geq \aleph_0$ ” is a first order property, Loś’ theorem (cf. [1]) gives that L is M_β for some infinite β . Moreover, $\beta \geq \alpha$, because if we define $f_j \in M_\alpha^I$ for each $j \in J$ by $f_j(i) = x_j$ for all $i \in I$, then the $[f_j]\mathcal{F}$ are mutually different. \square

We will use the following assertion, which is a consequence of 6.1.14 and 6.3.21 of [1].

Theorem 2.5. *Let I be any infinite set of the cardinality λ , A a set of the cardinality α . Then there exists an ultrafilter \mathcal{F} over I such that $\text{card } A^I/\mathcal{F} = \alpha^\lambda$.*

As a consequence we obtain

Theorem 2.6. *For each infinite cardinal α there exists a cardinal $\beta > \alpha$ with $M_\beta \in HCP\{M_\alpha\}$.*

PROOF. Take any set I of the cardinality α and an ultrafilter \mathcal{F} over I with $\text{card } M_\alpha^I/\mathcal{F} = \alpha^\alpha$. Set $\beta = \alpha^\alpha$. Then evidently $\beta > \alpha$ and, in view of 2.4, M_α^I/\mathcal{F} is isomorphic to M_β . \square

Corollary 2.7. *For each infinite cardinal α there exists an increasing infinite sequence of cardinals $\alpha_0 < \alpha_1 < \dots$ such that $\alpha_0 = \alpha$ and $HCP\{M_{\alpha_0}\} \supsetneq HCP\{M_{\alpha_1}\} \supsetneq HCP\{M_{\alpha_2}\} \supsetneq \dots$*

3. Convexities generated by finite chains

We will consider principal convexities generated by finite and also by some infinite chains and study relations between them.

Theorem 3.1. *For each $n \in \mathbb{N}$, $HCP\{C_n\} \subsetneq HCP\{C_{n+1}\}$.*

PROOF. Since C_{n+1} contains an n -element chain as a convex sublattice, it holds $HCP\{C_n\} \subseteq HCP\{C_{n+1}\}$. So we have only to show that $C_{n+1} \notin HCP\{C_n\}$ for each $n \in \mathbb{N}$. By way of contradiction, let n_0 be the least positive integer with $C_{n_0+1} \in HCP\{C_{n_0}\}$. Evidently $n_0 \geq 3$, because $HCP\{C_1\}$ contains only one-element lattices and each $L \in HCP\{C_2\}$ is a relatively complemented lattice, while C_3 fails to have this property. The relation $C_{n_0+1} \in HCP\{C_{n_0}\}$ implies that there exist an index set I , a convex sublattice B of $C_{n_0}^I$ and a homomorphism φ of B onto C_{n_0+1} . As C_{n_0+1} is bounded, we can suppose that B is an interval, say $[f_0, f_1]$ and $B = \prod(C_{k_i} : i \in J)$ with $J \subseteq I$, $1 < k_i \leq n_0$. Let us define $f \in B$ in such a way that, for $i \in J$, $f(i)$ is the least element of C_{k_i} if $k_i < n_0$ and the element covering the least one otherwise. Assume that C_{n_0+1} is the chain $c_0 < c_1 < \dots < c_{n_0}$, $\varphi(f) = c_t$. Then we have $\varphi([f_0, f]) = [c_0, c_t]$, $\varphi([f, f_1]) = [c_t, c_{n_0}]$. As $[f_0, f]$ is a convex sublattice of a product of two-element chains, it must be $t \leq 1$. On the other hand, $[f, f_1]$ is a convex sublattice of a product of $(n_0 - 1)$ -element chains, so that $t > 1$ by the choice of n_0 . We have a contradiction. \square

Corollary 3.2. *If C is any infinite chain, then $HCP\{C_n\} \subsetneq HCP\{C\}$ for each $n \in \mathbb{N}$.*

PROOF. Using an $(n - 1)$ -element subchain of C one can easily define a surjective homomorphism from C to C_n . The rest follows from 3.1. \square

Theorem 3.3. *It is $HCP\{C_n : n \in \mathbb{N}\} = HCP\{\mathbb{Z}\}$.*

PROOF. It suffices to show that $\mathbb{Z} \in HCP\{C_n : n \in \mathbb{N}\}$. Let \mathcal{F} be a nontrivial ultrafilter over \mathbb{N} and let $B = \prod(C_n \mid n \in \mathbb{N})/\mathcal{F}$. We will embed \mathbb{Z} into B . For each $n \in \mathbb{N}$ the first order property “there are at most $n + 1$ distinct x with $\| [0, x] \| \leq n$ ” holds in every C_i . Applying Loś’ theorem to this property and its dual we conclude that the set $U = \{x \in B : [0, x] \text{ is finite or } [x, 1] \text{ is finite}\}$ is countable (0 and 1 being the least and the greatest element of B , respectively). As B has the power of continuum, $B' = B - U$ is nonempty. Fix an element $b_0 \in B'$. Loś’ theorem again easily gives that for any $x \in B' - \{0, 1\}$ there is a unique upper resp. lower cover $x^+ \in B' - \{0, 1\}$ resp. $x^- \in B' - \{0, 1\}$ of x . Via induction we define $b_{n+1} = b_n^+$, $b_{-n-1} = b_{-n}^-$. Now it is easy to see that $\{b_n : n \in \mathbb{Z}\}$ is a convex sublattice of B , which completes the proof. \square

Proposition 3.4. *$HCP\{\mathbb{N}\} = HCP\{\mathbb{Z}\} = HCP\{\mathbb{R}\}$.*

PROOF. The first \subseteq is evident. The second \subseteq and the first \supseteq come from 3.2 and 3.3. Finally, $HCP\{\mathbb{Z}\} \supseteq HCP\{\mathbb{R}\}$ is a consequence of 1.2 of [7] where $\mathbb{R} \in HCP\{\mathbb{Z}\}$ is proved for a richer structure, namely for lattice ordered groups. \square

Let us remark that, as it was shown in [7], the convexity of l -groups generated by \mathbb{Z} is larger than that generated by \mathbb{R} .

4. An example

J. JAKUBÍK proved in [4] that the convexity $HCP\{C_2\}$ is an atom in the lattice \mathcal{C} of all convexities of lattices. He also formulated the question if there are other atoms in \mathcal{C} . This question remains open. We give here some results concerning this problem. Further we prove that the lattice \mathcal{C} is distributive.

Let L be a lattice. Consider the following conditions concerning L :

- (i) L contains a non-trivial distributive interval;
- (ii) L has a non-trivial distributive homomorphic image.

Theorem 4.1. *Let L be a lattice satisfying any of the conditions (i), (ii). Then $HCP\{L\} \supseteq HCP\{C_2\}$.*

PROOF. Without regard to which of the conditions (i), (ii) is fulfilled, the convexity $HCP\{L\}$ contains a distributive lattice L_1 with $\text{card } L_1 > 1$. By a well-known theorem each distributive lattice containing more than one element can be homomorphically mapped onto C_2 . So $HCP\{C_2\} \subseteq HCP\{L_1\} \subseteq HCP\{L\}$. \square

Let us remark that the condition (i) is fulfilled, e.g., by each finite lattice or, more generally, by each lattice containing a prime interval. We are going to show that the converse assertion to 4.1 does not hold in general.

If P is any partially ordered set and S is a bounded partially ordered set, then we will use the notation $(S \rightarrow P)$ for the partially ordered set obtained in such a way that each prime interval of P is replaced by S . For example, $(C_3 \rightarrow (C_2 \times C_2))$ is an 8-element lattice consisting of two copies of C_5 having the same endpoints. It is easy to see that if S, P are finite lattices, then so is $(S \rightarrow P)$ and P can be regarded as its sublattice.

Now take the lattice M_3 and define $L_i (i \in N_0)$ as follows: $L_0 = M_3$; if L_i is defined for some $i \in N_0$, then $L_{i+1} = (M_3 \rightarrow L_i)$.

We have an ascending chain of lattices $L_0 \subseteq L_1 \subseteq L_2 \subseteq \dots$ with L_i being a sublattice of L_{i+1} for each $i \in N_0$. Let L be the join of all L_i . Evidently L is a

lattice and each L_i is its sublattice. It is easy to see that L is not even modular. The following lemma shows that L does not satisfy (ii).

Lemma 4.2. *The lattice L has only trivial congruence relations.*

PROOF. An easy induction shows that all the L_i are simple, whence so is L . \square

To show that L doesn't satisfy (i), let us look at the intervals of L .

Lemma 4.3. *Each non-trivial interval of L contains a convex sublattice isomorphic to L .*

PROOF. The obvious induction is left to the reader. \square

Using 4.3 we obtain immediately that L does not satisfy (i), because L is not distributive, as we have already remarked. In spite of the fact that L satisfies neither (i) nor (ii), we will show that $HCP\{L\} \supseteq HCP\{C_2\}$.

Let us define a sequence a_0, a_1, a_2, \dots of elements of L as follows: a_0 will be the least element of L_0 . Take any $b \in L_0$ covering a_0 in L_0 and denote by a_1 any element of L_1 which lies between a_0 and b . Further denote by a_2 any element of L_2 which lies between a_1 and b , and so on.

We have $a_0 < a_1 < \dots < b$, and for each $i \in N$, $a_i \in L_i - L_{i-1}$, a_i covers a_{i-1} in L_i . Introduce the following notations: for each $i \in N$, I_i will be the set $\{x \in L : a_0 \leq x \leq a_i\}$, $M = \prod (I_i \mid i \in N)$. It is easy to see that I_i is isomorphic to $(L \rightarrow C_{i+1})$ (see Figure 1).

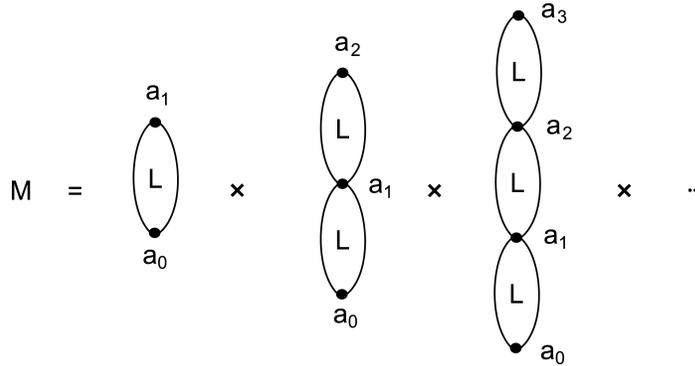


Figure 1

Theorem 4.4. *Let L be the lattice defined before 4.2. Then $HCP\{C_2\} \subsetneq HCP\{L\}$.*

PROOF. Let \mathcal{F} be a nontrivial ultrafilter over \mathbb{N} and define P to be the set $\{f \in M : \text{there is an } n \in \mathbb{N} \text{ with } \{i : f(i) \leq a_n\} \in \mathcal{F}\}$. Of course, here and in the sequel, $f(i) \leq a_n$ is understood to be true when $i < n$ (i.e., when $a_n \notin I_i$). Clearly, P is an ideal of the lattice M , $\emptyset \neq P \neq M$. In order to show that P is a prime ideal, let us assume that for $f_1, f_2 \in M$ we have $g = f_1 \wedge f_2 \in P$. Then there is an $n \in \mathbb{N}$ such that $U = \{i : g(i) \leq a_n\} \in \mathcal{F}$. The structure of the I_i makes it clear that $g(i) \leq a_n$ implies $f_1(i) \leq a_{n+1}$ or $f_2(i) \leq a_{n+1}$. Hence, with $V_j = \{i : f_j(i) \leq a_{n+1}\}$ for $j \in \{1, 2\}$, we have $U \subseteq V_1 \cup V_2$. Since \mathcal{F} is an ultrafilter, we derive $V_1 \in \mathcal{F}$ or $V_2 \in \mathcal{F}$, i.e. $f_1 \in P$ or $f_2 \in P$. This shows that P is a prime ideal of M , whence the two-element lattice is a homomorphic image of M . Now the I_i , their direct product M , and therefore C_2 , belong to $HCP\{L\}$. Finally $L \notin HCP\{C_2\}$, because $HCP\{C_2\}$ contains only distributive lattices. \square

J. JAKUBÍK proved in [7] that the lattice of all convexities of l -groups is distributive. This result was extended to the lattice of all convexities of Riesz groups in [8]. Now we are going to prove the distributivity of the lattice \mathcal{C} of all convexities of lattices.

Lemma 4.5. *Let $\mathcal{K}_1, \mathcal{K}_2$ be convexities of lattices. Then $\mathcal{K} = \{L \times M : L \in \mathcal{K}_1, M \in \mathcal{K}_2\}$ is a convexity and it holds $\mathcal{K} = \mathcal{K}_1 \vee \mathcal{K}_2$.*

PROOF. The inclusion $\mathcal{K} \subseteq \mathcal{K}_1 \vee \mathcal{K}_2$ is trivial. Let $A \in \mathcal{K}_1 \vee \mathcal{K}_2 = HCP(\mathcal{K}_1 \cup \mathcal{K}_2)$. Then there exist lattices $L_i \in \mathcal{K}_1 \cup \mathcal{K}_2 (i \in I, I \neq \emptyset)$, a convex sublattice B of $\prod(L_i \mid i \in I)$ and a congruence relation ϕ of B such that A is isomorphic to B/ϕ . Let $I_1 = \{i \in I : L_i \in \mathcal{K}_1\}$, $U = \prod(L_i \mid i \in I_1)$, $V = \prod(L_i \mid i \in I - I_1)$. If some of the sets $I_1, I - I_1$ is empty, the corresponding direct product is regarded as a one-element lattice. We can suppose that B is a convex sublattice of $U \times V$. Let us denote by B_1 and B_2 the projection of B into U and V , respectively. It is easy to verify that B_1 and B_2 is a convex sublattice of U and V , respectively, and $B = B_1 \times B_2$. Now there exist congruence relations ϕ_1 of B_1 and ϕ_2 of B_2 with $\phi = \phi_1 \times \phi_2$ (cf. [2]). Then B/ϕ is isomorphic to $B_1/\phi_1 \times B_2/\phi_2$ and $B_1/\phi_1 \in \mathcal{K}_1$, $B_2/\phi_2 \in \mathcal{K}_2$. Hence A , being isomorphic to B/ϕ , belongs to \mathcal{K} . \square

Theorem 4.6. *The lattice \mathcal{C} of all convexities of lattices is distributive.*

PROOF. Let $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3 \in \mathcal{C}$. We are going to verify $\mathcal{K}_1 \wedge (\mathcal{K}_2 \vee \mathcal{K}_3) = (\mathcal{K}_1 \wedge \mathcal{K}_2) \vee (\mathcal{K}_1 \wedge \mathcal{K}_3)$. Clearly $\mathcal{K}_1 \wedge (\mathcal{K}_2 \vee \mathcal{K}_3) \supseteq (\mathcal{K}_1 \wedge \mathcal{K}_2) \vee (\mathcal{K}_1 \wedge \mathcal{K}_3)$. Now let $L \in \mathcal{K}_1 \wedge (\mathcal{K}_2 \vee \mathcal{K}_3)$. Then $L \in \mathcal{K}_1$ and L is isomorphic to $L_1 \times L_2$ for some $L_1 \in \mathcal{K}_2, L_2 \in \mathcal{K}_3$ by 4.5. We can suppose that L_1, L_2 are convex sublattices of L , so that $L_1, L_2 \in CK_1 = \mathcal{K}_1$. Hence we have $L \in (\mathcal{K}_1 \wedge \mathcal{K}_2) \vee (\mathcal{K}_1 \wedge \mathcal{K}_3)$. \square

Open problems:

1. Is $HCP\{C_2\}$ the only atom in \mathcal{C} ? (formulated in [4])
2. What are the necessary and sufficient conditions for a distributive relatively complemented lattice L to belong to $HCP\{C_2\}$?
3. What are the necessary and sufficient conditions for a distributive lattice L to belong to $HCP\{C_3\}$?
4. Does the convexity $HCP\{C_{n+1}\}$ cover the convexity $HCP\{C_n\}$ for $n \in \mathbb{N}$?
5. Does the convexity $HCP\{M_n\}$ ($n \in \mathbb{N}$, $n \geq 3$) cover the convexity $HCP\{C_2\}$?

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References

- [1] J. L. BELL and A. B. SLOMSON, Models and Ultraproducts, *North Holland Publishing Company, Amsterdam, London*, 1969.
- [2] G. A. FRASER and A. HORN, Congruence relations in direct products, *Proc. Amer. Math. Soc.* **26** (1970), 390–394.
- [3] G. GRÄTZER, General Lattice Theory, *Birkhäuser Verlag*, 1998.
- [4] J. JAKUBÍK, On convexities of lattices, *Czechoslovak Math. J.* **42**(117) (1992), 325–330.
- [5] J. JAKUBÍK, On convexities of d -groups, *Czechoslovak Math. J.* **44**(119) (1994), 305–314.
- [6] J. JAKUBÍK, Convexities of lattice ordered groups, *Math. Bohem.* **121**, no. 1 (1996), 59–67.
- [7] J. JAKUBÍK, Convexities of normal valued lattice ordered groups, *Czechoslovak Math. J.* **50**(125) (2000), 641–649.
- [8] J. LIHOVÁ, Convexities of Riesz groups, *Tatra Mt. Math. Publ.* **30** (2005), 71–85.
- [9] R. MLITZ (editor), General Algebra, Proc. Internat. Conf. Krems, 1988, *North Holland, Amsterdam – New York – Tokyo – Oxford*, 1990.

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