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The $\mathcal{L}$-dual of a Matsumoto space

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Abstract. In [HS1], [MHSS] the $\mathcal{L}$-duals of a Randers and Kropina space were studied. In this paper we shall discuss the $\mathcal{L}$-dual of a Matsumoto space. The metric of this $\mathcal{L}$-dual space is completely new and it brings a new idea about $\mathcal{L}$-duality because the $\mathcal{L}$-dual of Matsumoto metric can be given by means of four quadratic forms and 1-forms on T^*M constructed only with the Riemannian metric coefficients, $a_{ij}(x)$ and the 1-form coefficients $b_i(x)$.

1. Introduction

The study of $\mathcal{L}$-duality of Lagrange and Finsler space was initiated by R. Miron [Mi2] around 1980. Since then, many Finsler geometers studied this topic.

One of the remarkable results obtained are the concrete $\mathcal{L}$-duals of Randers and Kropina metrics [HS2]. However, the importance of $\mathcal{L}$-duality is by far limited to computing the dual of some Finsler fundamental functions.

Recently, in [BRS], the complicated problem of classifying Randers metrics of constant flag curvature was solved by means of duality. Other geometrical problems of (α,β)-metrics might be solved on future by considering not the metric itself, but its $\mathcal{L}$-dual.

The concrete examples of $\mathcal{L}$-dual metrics are quite few [HS1], [HS2]. In the present paper we succeeded to compute the dual of another well known (α,β)-metric, the Matsumoto metric. Surprisingly, despite of the quite complicated computations involved, we obtain the Hamiltonian function by means of four

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quadratic forms and a 1-form on T^*M. This metric is completely new and it brings a new idea about $\mathcal{L}$-duality. The dual of an (α,β)-metric can be given by means of several quadratic forms and 1-forms on T^*M constructed only with the Riemannian metric coefficients, $a_{ij}(x)$ and the 1-form coefficients $b_i(x)$.

2. The Legendre transformation

2.1. Definitions. Let $F^n = (M, F)$ be a n-dimensional Finsler space. The fundamental function $F(x,y)$ is called *an* (α,β)*-metric* if F is homogeneous function of α and β of degree one, where $\alpha^2 = a(y,y) = a_{ij}y^iy^j$, $y = y^i\frac{\partial}{\partial x^i}|_x \in T_xM$ is Riemannian metric, and $\beta = b_i(x)y^i$ is a 1-form on $\widetilde{TM} = TM \setminus \{0\}$.

A Finsler space with the fundamental function:

$$F(x,y) = \alpha(x,y) + \beta(x,y), \tag{2.1}$$

is called a *Randers space*.

A Finsler space having the fundamental function:

$$F(x,y) = \frac{\alpha^2(x,y)}{\beta(x,y)}, \tag{2.2}$$

is called a *Kropina space*, and one with

$$F(x,y) = \frac{\alpha^2(x,y)}{\alpha(x,y) - \beta(x,y)}, \tag{2.3}$$

is called a *Matsumoto space*.

Let $C^n = (M, K)$ be an n-dimensional Cartan space having the fundamental function $K(x,p)$. We also consider Cartan spaces having the metric function of the following form:

$$K(x,p) = \sqrt{a^{ij}(x)p_ip_j} + b^i(x)p_i, \tag{2.4}$$

or

$$K(x,p) = \frac{a^{ij}(x)p_ip_j}{b^i(x)p_i}, \tag{2.5}$$

with $a_{ij}a^{jk} = \delta_i^k$ and we will again call these spaces Randers and Kropina spaces on the cotangent bundle T^*M, respectively.

Let $L(x,y)$ be a regular Lagrangian on a domain $D \subset TM$ and let $H(x,p)$ be a regular Hamiltonian on a domain $D^* \subset T^*M$.

It is known [MHSS] that if L is a differentiable function, we can consider the fiber derivative of L, locally given by the diffeomorphism between the open set $U \subset D$ and $U^* \subset D^*$:

$$\varphi(x, y) = (x^i, \dot{\partial}_a L(x, y)) \tag{2.6}$$

which is called the *Legendre transformation.* We can define, in this case, the function $H : U^* \to R$:

$$H(x, p) = p_a y^a - L(x, y), \tag{2.7}$$

where $y = (y^a)$ is the solution of the equations:

$$p_a = \dot{\partial}_a L(x, y). \tag{2.8}$$

In the same manner, the fiber derivative is locally given by:

$$\psi(x, p) = (x^i, \dot{\partial}^a H(x, p)). \tag{2.9}$$

The function ψ is a diffeomorphism between the same open sets $U^* \subset D^*$ and $U \subset D$ and we can consider the function $L : U \to R$:

$$L(x, y) = p_a y^a - H(x, p), \tag{2.10}$$

where $p = (p_a)$ is the solution of the equations:

$$y^a = \dot{\partial}^a H(x, p). \tag{2.11}$$

The Hamiltonian given by (2.7) is called the *Legendre transformation of the Lagrangian L* and the Lagrangian given by (2.10) is called the *Legendre transformation of the Hamiltonian H.*

If (M, K) is a Cartan space, then (M, H) is a Hamilton manifold [MHSS], where $H(x, p) = \frac{1}{2} K^2(x, p)$ is 2-homogeneous on a domain of T^*M. So, we get the following transformation of H on U:

$$L(x, y) = p_a y^a - H(x, p) = H(x, p). \tag{2.12}$$

Proposition 1 ([MHSS])**.** *The scalar field $L(x, y)$ defined by* (2.12) *is a positively* 2*-homogeneous regular Lagrangian on U.*

Therefore, we get the Finsler metric F of U, such that

$$L = \frac{1}{2} F^2 \tag{2.13}$$

Thus, for the Cartan space (M, K) one always can locally associate a Finsler space (M, F) which will be called the *$\mathcal{L}$-dual of a Cartan space* $(M, K_{|U^*})$.

Conversely, we can associate, locally, a Cartan space to every Finsler space which will be called the *$\mathcal{L}$-dual of a Finsler space* $(M, F_{|U})$.

3. The (α,β) Finsler – (α^*,β^*) Cartan $\mathcal{L}$-duality

Let us recall some known results.

Theorem 3.1 ([HS1], [MHSS]). *Let (M,F) be a Randers space and $b=(a_{ij}b^ib^j)^{\frac{1}{2}}$ the Riemannian length of b_i. Then:*

(1) *If $b^2=1$, the $\mathcal{L}$-dual of (M,F) is a Kropina space on T^*M with:*

$$H(x,p)=\frac{1}{2}\left(\frac{a^{ij}p_ip_j}{2b^ip_i}\right)^2. \tag{3.1}$$

(2) *If $b^2\neq 1$, the $\mathcal{L}$-dual of (M,F) is a Randers space on T^*M with:*

$$H(x,p)=\frac{1}{2}\left(\sqrt{\tilde{a}^{ij}p_ip_j}\pm\tilde{b}^ip_i\right)^2, \tag{3.2}$$

where

$$\tilde{a}^{ij}=\frac{1}{1-b^2}a^{ij}+\frac{1}{(1-b^2)^2}b^ib^j;\quad \tilde{b}^i=\frac{1}{1-b^2}b^i,$$

(in (3.2) $'-'$ corresponds to $b^2<1$ and $'+'$ corresponds to $b^2>1$).

Theorem 3.2 ([HS1], [MHSS]). *The $\mathcal{L}$-dual of a Kropina space is a Randers space on T^*M with the Hamiltonian:*

$$H(x,p)=\frac{1}{2}\left(\sqrt{\tilde{a}^{ij}p_ip_j}\pm\tilde{b}^ip_i\right)^2, \tag{3.3}$$

where

$$\tilde{a}^{ij}=\frac{b^2}{4}a^{ij};\quad \tilde{b}^i=\frac{1}{2}b^i,$$

(in (3.3) $'-'$ corresponds to $\beta<0$ and $'+'$ corresponds to $\beta>0$).

In [HS1] the notation $\alpha^*=(a^{ij}(x)p_ip_j)^{\frac{1}{2}}$, $\beta^*=b^i(x)p_i$ are used, where $a^{ij}(x)$ are the reciprocal components of $a_{ij}(x)$ and $b^i(x)$ are the components of the vector field on M, $b^i(x)=a^{ij}(x)b_j(x)$. We can consider the metric functions $K=\alpha^*+\beta^*$ (Randers metric on T^*M) or $K=\frac{\alpha^{*2}}{\beta^*}$ (Kropina metric on T^*M) defined on a domain $D^*\subset T^*M$. So, one can easily rewrite the previous theorems:

Theorem 3.3. *Let (M,F) be a Randers space and $b=(a_{ij}b^ib^j)^{\frac{1}{2}}$ the Riemannian lengh of b_i. Then:*

(1) *If* $b^2 = 1$, *the* $\mathcal{L}$*-dual of* (M, F) *is a Kropina space on* T^*M *with:*

$$H(x,p) = \frac{1}{2}\left(\frac{\alpha^{*2}}{2\beta^*}\right)^2. \tag{3.4}$$

(2) *If* $b^2 \neq 1$, *the* $\mathcal{L}$*-dual of* (M, F) *is a Randers space on* T^*M *with:*

$$H(x,p) = \frac{1}{2}\left(\alpha^* \pm \beta^*\right)^2, \tag{3.5}$$

with $\alpha^* = \sqrt{\tilde{a}^{ij}(x)p_ip_j}$ *and* $\beta^* = \tilde{b}^i p_i$ *where*

$$\tilde{a}^{ij} = \frac{1}{1-b^2}a^{ij} + \frac{1}{(1-b^2)^2}b^ib^j; \quad \tilde{b}^i = \frac{1}{1-b^2}b^i,$$

(in (3.5) $'-'$ *corresponds to* $b^2 < 1$ *and* $'+'$ *corresponds to* $b^2 > 1$*).*

Theorem 3.4. *The* $\mathcal{L}$*-dual of a Kropina space is a Randers space on* T^*M *with the Hamiltonian:*

$$H(x,p) = \frac{1}{2}\left(\alpha^* \pm \beta^*\right)^2, \tag{3.6}$$

with $\alpha^* = \sqrt{\tilde{a}^{ij}(x)p_ip_j}$ *and* $\beta^* = \tilde{b}^i p_i$ *where*

$$\tilde{a}^{ij} = \frac{b^2}{4}a^{ij}; \quad \tilde{b}^i = \frac{1}{2}b^i,$$

(in (3.6) $'-'$ *corresponds to* $\beta < 0$ *and* $'+'$ *corresponds to* $\beta > 0$*).*

We are going to compute now the dual of a Matsumoto metric. We obtain:

Theorem 3.5. *Let* (M, F) *be a Matsumoto space and* $b = (a_{ij}b^ib^j)^{\frac{1}{2}}$ *the Riemannian length of* b_i*. Then*

(1) *If* $b^2 = 1$, *the* $\mathcal{L}$*-dual of* (M, F) *is the space having the fundamental function:*

$$H(x,p) = \frac{1}{2}\left(-\frac{b^ip_i}{2}\frac{\left(\sqrt[3]{a^{ij}p_ip_j} + \sqrt[3]{\left(b^ip_i + \sqrt{\tilde{a}^{ij}p_ip_j}\,\right)^2}\,\right)^3}{a^{ij}p_ip_j + \left(b^ip_i + \sqrt{\tilde{a}^{ij}p_ip_j}\right)^2}\right)^2, \tag{3.7}$$

where

$$\tilde{a}^{ij} = b^ib^j - a^{ij}.$$

(2) *If $b^2 \neq 1$, the $\mathcal{L}$-dual of (M,F) is the space on T^*M having the fundamental function:*

$$H(x,p)=\frac{1}{2}\left(-\frac{b^ip_i}{200}\frac{25\left(2\sqrt{d_2^{ij}p_ip_j}+\sqrt{d_4^{ij}p_ip_j}\right)^2+d_8^{ij}p_ip_j}{\sqrt{d_2^{ij}p_ip_j}\sqrt{d_4^{ij}p_ip_j}+d_9^{ij}p_ip_j}\right)^2, \tag{3.8}$$

where

$$\begin{aligned}
c_1^{ij}&=(b^ib^j+2\varepsilon_1a^{ij})^2+(2a^{ij})^2\varepsilon_3,\\
c_2^{ij}&=a^{ij}(\theta_4^2b^ib^j+a^{ij}\varepsilon_2),\\
c_3^{ij}&=(2a^{ij})^2\theta_5^3,\\
\sqrt[3]{\tilde{a}^{ij}}^2&=\sqrt[3]{c_1^{ij}}-2\sqrt[3]{c_2^{ij}}+\sqrt[3]{c_3^{ij}},\\
d_1^{ij}&=d_3^{ij}+4m(a^{ij}b^2-b^ib^j),\\
d_2^{ij}&=\sqrt{d_3^{ij}a^{ij}}+4\sqrt{d_1^{ij}a^{ij}}-d_3^{ij},\\
d_3^{ij}&=2\sqrt[3]{2a^{ij}(\tilde{a}^{ij})^2},\\
\sqrt{d_4^{ij}}&=\sqrt{d_3^{ij}}+3\sqrt{a^{ij}},\\
\sqrt{d_5^{ij}}&=\sqrt{d_3^{ij}a^{ij}},\\
d_6^{ij}&=d_1^{ij}a^{ij},\\
\sqrt{d_7^{ij}}&=2\sqrt{d_2^{ij}}+\sqrt{d_4^{ij}},\\
d_8^{ij}&=200\left(\sqrt{d_6^{ij}}+2na^{ij}\right)-5\left(4\sqrt{d_3^{ij}}+\sqrt{d_4^{ij}}\right),\\
d_9^{ij}&=4\sqrt{d_6^{ij}}+4a^{ij}p+9\sqrt{d_5^{ij}},
\end{aligned}$$

and

$$\begin{aligned}
m&=1-b^2,\\
n&=\frac{20b^2-29}{29},\\
p&=\frac{1-2b^2}{2},\\
\theta_1&=-\frac{712b^6-452b^4+24b^2+1}{1728},\\
\theta_2&=\frac{576b^4-2232b^2+2628}{1728},
\end{aligned}$$

$$
\begin{aligned}
\theta_3 &= -\left(\frac{8b^2+1}{12}\right)^2,\\
\theta_4 &= \frac{2b^2+1}{6},\\
\theta_5 &= \frac{11b^2+1}{12},\\
\varepsilon_1 &= 2(\theta_4^2-\theta_2),\\
\varepsilon_2 &= 3\theta_3\theta_4^2+\theta_2^2,\\
\varepsilon_3 &= 4\varepsilon_2-2\theta_1-\varepsilon_1.
\end{aligned}
$$

Proof. By putting: $\alpha^2 = y_i y^i$, $b^i = a^{ij} b_j$, $\beta = b_i y^i$, $\beta^* = b^i p_i$, $p^i = a^{ij} p_j$, $\alpha^{*2} = p_i p^i = a^{ij} p_i p_j$, we have $F = \frac{\alpha^2}{\alpha-\beta}$, and

$$
p_i = \frac{1}{2}\dot{\partial}_i F^2 = \frac{y_i}{\alpha-\beta} + \frac{\alpha^2 b^i - y_i\beta}{(\alpha-\beta)^2}. \tag{3.9}
$$

Contracting in (3.9) by p^i and b^i we get:

$$
\begin{aligned}
\alpha^{*2} &= \frac{F}{(\alpha-\beta)^2}[F^2(\alpha-2\beta)+\alpha^2\beta^*]\\
\beta^* &= \frac{F}{(\alpha-\beta)^2}[\beta(\alpha-2\beta)+\alpha^2 b^2].
\end{aligned} \tag{3.10}
$$

In [Sh], for a Finsler (α,β)-metric F on a manifold M, one constructs a positive function $\phi = \phi(s)$ on $(-b_0; b_0)$ with $\phi(0) = 1$ and $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij} y^i y^j}$ and $\beta = b_i y^i$ with $\|\beta\|_x < b_0$, $\forall x \in M$.

The function ϕ satisfies: $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0$, $(|s| \le b_0)$.

A Matsumoto metric is a special (α,β)-metric with $\phi = \frac{1}{1-s}$.

Using Shen's [Sh] notation $s = \frac{\beta}{\alpha}$, the formula (3.10) become:

$$
\begin{aligned}
\alpha^{\star 2} &= F^2\frac{1-2s}{(1-s)^3} + F\frac{1}{(1-s)^2}\beta^\star,\\
\beta^\star &= Fs\frac{1-2s}{(1-s)^2} + F\frac{1}{(1-s)^2}b^2.
\end{aligned} \tag{3.11}
$$

Now we put $1-s=t$, i.e. $s = 1-t$ and both equations become:

$$
\alpha^{\star 2} = F^2\frac{2t-1}{t^3} + F\frac{1}{t^2}\beta^\star, \tag{3.12}
$$

$$\beta^\star = F(1-t)\frac{2t-1}{t^2} + F\frac{1}{t^2}b^2. \tag{3.13}$$

We get

$$\beta^* t^2 = M(-2t^2 + 3t + b^2 - 1). \tag{3.14}$$

For $b^2 = 1$ from (3.13) we obtain:

$$F = -\frac{\beta^* t}{2t-3}, \tag{3.15}$$

and by substitution of F in (3.12), after some computations we get a cubic equation:

$$t^3 - 3t + \frac{9}{4}t - \frac{\beta^\star}{2\alpha^{\star 2}} = 0. \tag{3.16}$$

Using Cardano's method for solving cubic equation [Wi], we get:

$$F = -\frac{\beta^\star}{2}\frac{(2P-1)^2}{3P^2 + (P-1)^2}, \tag{3.17}$$

where for P we have:

$$P = \frac{1}{2}\sqrt[3]{\left(\frac{\beta^\star + \sqrt{\beta^{\star 2} - \alpha^{\star 2}}}{\alpha^\star}\right)^2}. \tag{3.18}$$

After some computations, for F we get:

$$F = -\frac{\beta^\star}{2}\frac{\left(\sqrt[3]{\alpha^{\star 2}} + \sqrt[3]{(\beta^\star + \sqrt{\beta^{\star 2} - \alpha^{\star 2}}\,)^2}\right)^3}{\alpha^{\star 2} + (\beta^\star + \sqrt{\beta^{\star 2} - \alpha^{\star 2}})^2}. \tag{3.19}$$

Substituting now $\beta^* = b^i p_i$ and $\alpha^{*2} = p_i p^i = a^{ij} p_i p_j$ we can easily get (3.7).

If $b^2 \neq 1$, the formula (3.15) is more complicated because:

$$F = \frac{\beta^* t^2}{-2t^2 + 3t + b^2 - 1}, \tag{3.20}$$

and by substituting this in (3.12) we obtain the quadric equation:

$$t^4 - 3t^3 + t^2\frac{13 - 4b^2}{4} + t\frac{6\alpha^{*2}(b^2-1)}{4\alpha^{*2}} + \frac{\alpha^{*2}(b^2-1)^2 + \beta^{*2}(1-b^2)}{4\alpha^{*2}} = 0. \tag{3.21}$$

After a quite long computation, formula (3.21) becomes a cubic equation (different from (3.16)) and solving it, we get:

$$\begin{aligned} F = -\frac{\beta^*}{2}\Bigg(\bigg(\sqrt{-A^2+3A+2\sqrt{A^2+m\Big(b^2-\frac{\beta^{*2}}{\alpha^{*2}}\Big)}}+\frac{A}{2}+\frac{3}{4}\bigg)^2 \\ +\sqrt{A^2+m\Big(b^2-\frac{\beta^{*2}}{\alpha^{*2}}\Big)}-\frac{5}{4}\Big(A+\frac{3}{10}\Big)^2+n\Bigg) \\ \Bigg/\Bigg(\Big(\frac{3}{2}+2A\Big)(\sqrt{-A^2+3A+2\sqrt{A^2+m\Big(b^2-\frac{\beta^{*2}}{\alpha^{*2}}\Big)}} \\ +2\sqrt{A^2+m\Big(b^2-\frac{\beta^{*2}}{\alpha^{*2}}\Big)}+\frac{9}{2}A+p\Bigg), \end{aligned} \tag{3.22}$$

where

$$A^2 = \sqrt[3]{\Big(\frac{1}{2}\frac{\beta^{*2}}{\alpha^{*2}}+\varepsilon_1\Big)^2+\varepsilon_3}+\sqrt[3]{-4\Big(\theta_4^3\frac{\beta^{*2}}{\alpha^{*2}}+\varepsilon_2\Big)}+\theta_5. \tag{3.23}$$

By substituting now $\beta^* = b^i p_i$ and $\alpha^{*2} = p_i p^i = a^{ij} p_i p_j$, after some computations, from (3.23) we obtain (3.8). □

3.1. Remarks.

(1) It is easy to see that both relations, (3.7) and (3.8), are coming from (3.14). Indeed, substituting $b^2 = 1$ in (3.14) we get the cubic equation (3.16). As solution, we find (3.7). For $b^2 \neq 1$, from (3.14) we get the complicated quadric equation (3.21) with (3.8) as solution. If in (3.21) we would replace $b^2 = 1$ we would get $t^4 - 3t^3 + \frac{9}{4} = 0$ with $t_1 = t_2 = 0$ and $t_3 = t_4 = \frac{3}{2}$. It is impossible for these four solutions to exist in our proof. So, we can easily see that (3.7) and (3.8) are two different relations and we can't get (3.7) as a particular case of (3.8).

(2) Using α^* and β^* we can get, for the $\mathcal{L}$-dual of (M, F), in the case $b^2 = 1$, the fundamental function:

$$H(x,p) = \frac{1}{2}\left(-\frac{\beta^\star}{2}\frac{\Big(\sqrt[3]{\alpha^{\star 2}}+\sqrt[3]{\big(\beta^\star+\sqrt{\beta^{\star 2}-\alpha^{\star 2}}\big)^2}\Big)^3}{\alpha^{\star 2}+(\beta^\star+\sqrt{\beta^{\star 2}-\alpha^{\star 2}})^2}\right)^2. \tag{3.24}$$

(3) In (3.7) $\tilde{a}^{ij}$ is positive-definite and the Randers metric on T^*M $p_i b^i + \sqrt{p_i p_j \tilde{a}^{ij}}$ is positive-valued for any p.

4. Conclusions

Let's take a second look at formula (3.8). If we introduce the following quadratic forms:

$$\alpha_2^* = \sqrt{d_2^{ij} p_i p_j}, \qquad \alpha_4^* = \sqrt{d_4^{ij} p_i p_j},$$
$$\alpha_8^* = \sqrt{d_8^{ij} p_i p_j}, \qquad \alpha_9^* = \sqrt{d_9^{ij} p_i p_j},$$

defined on T^*M by the corresponding matrices, then (3.8) becomes:

$$H(x,p) = \frac{1}{2}\left(-\frac{\beta^*}{200} \frac{25\left(2\alpha_2^* + \alpha_4^*\right)^2 + \left(\alpha_8^*\right)^2}{\alpha_2^*\alpha_4^* + \left(\alpha_9^*\right)^2} \right)^2, \tag{4.1}$$

for $b^2 \neq 1$.

In other words, the $\mathcal{L}$-duals of a Randers and Kropina metrics are expressed only with the duals α^*, β^* of α, β, respectively. However, the $\mathcal{L}$-dual of a Matsumoto metric is given by means of four distinct quadratic forms on T^*M. Remark that the coefficients of the quadratic forms are constructed only from the Riemannian metric matrix element, a_{ij} and the 1-forms β's coefficients $b_i(x)$.

Inevitably, the following question occurs: if d_2^{ij}, d_4^{ij}, d_8^{ij}, d_9^{ij} are positively defined and therefore making sure that α_2^*, α_4^*, α_8^*, α_9^* exist.

The answer is not quite immediate and depends both on the value of b^2 and on a^{ij}, b^i, b^j. For example, if we take $b^2 < \frac{1}{2}$ and $a^{ij} > 2b^i b^j$ then, not only d_2^{ij}, d_4^{ij}, d_8^{ij}, d_9^{ij} are positively defined but also the four quadric forms are defined.

Certainly, there are many other values for b^2, a^{ij}, b^i, b^j which give a certain positive answer, but the above values justify the existence of (4.1).

4.1. Remarks, examples.

Remark 4.1. For the $\mathcal{L}$-dual of (4.1) we obtain the Matsumoto space with the fumdamental function:

$$F = \frac{\tilde{a}_{ij} y^i y^j}{\sqrt{b^2 a_{ij} y^i y^j} - \tilde{b}_i y^i}, \tag{4.2}$$

where

$$\begin{aligned} \tilde{b}_i &= 4b^2 b_i, \\ \tilde{a}_{ij} &= a_{ij}^2 b_i b_j (7 + 8b^2) - \sqrt{a_{ij}}\, b_i [a_{ij}(1 + 2b^2) - 12 b_i b_j] \\ &\quad \pm m[a_{ij}^2 b_i (7 + 8b^2) - \sqrt{a_{ij}}(a_{ij} - 12 b_i b_j)], \end{aligned}$$

and

$$m = \sqrt{b_i b_j - b^2 a_{ij}}.$$

The other properties like curvature and the relation between geometrical properties of the $\mathcal{L}$-dual metric (4.1) and the initial Matsumoto metric will be studied elsewhere.

Example 1. Let us consider a particular example and find its $\mathcal{L}$-dual. For this, let us consider a surface S emebedded in the usual Euclidian space R^3, i.e.

$$S \hookrightarrow R^3, \quad (x, y) \in S \longrightarrow (x, y, z = f(x, y)) \in R^3.$$

It is known that the induced Riemannian metric on the surface S is given by:

$$(a_{ij}) = \begin{pmatrix} 1 + (f_x)^2 & f_x f_y \\ f_x f_y & 1 + (f_y)^2 \end{pmatrix},$$

where f_x and f_y means partial derivative with respect to x and y, respectively.

If we consider now a coordinate system $(x, y, u, v) \in TM$ in the tangent bundle TM, then for α and β one can choose:

$$\alpha^2 = (1 + f_x^2)^2 u^2 + 2 f_x f_y uv + (1 + f_y^2)^2 v^2,$$

and

$$\beta = f_x u + f_y v.$$

Now, for the induced Riemannian metric, we have:

$$\det \|a_{ij}\| = 1 + f_x^2 + f_y^2,$$

$$(a^{ij}) = \begin{pmatrix} \dfrac{1 + (f_y)^2}{1 + f_x^2 + f_y^2} & -\dfrac{f_x f_y}{1 + f_x^2 + f_y^2} \\ -\dfrac{f_x f_y}{1 + f_x^2 + f_y^2} & \dfrac{1 + (f_x)^2}{1 + f_x^2 + f_y^2} \end{pmatrix},$$

$$\tilde{b}^1 = \frac{f_x}{1 + f_x^2 + f_y^2}, \quad \tilde{b}^2 = \frac{f_y}{1 + f_x^2 + f_y^2},$$

and for the Riemannian length of $\tilde{b}_i$:

$$b^2 = \frac{f_x^2 + f_y^2}{1 + f_x^2 + f_y^2}, \quad 0 < b^2 < 1.$$

Using these and following step by step the second case of Theorem 3.5, we find:

$$d_2^{11} = M(A + 4B) - A^2,$$

$$d_2^{12} = d_2^{21} = P^2[E(1-E) + 4F],$$
$$d_2^{22} = N(C + 4D) - C^2,$$
$$d_4^{11} = A + 3M,$$
$$d_4^{12} = d_4^{21} = P(E + 3),$$
$$d_4^{22} = C + 3N,$$
$$d_8^{11} = 5M[40(B + 2nM) - 3] - 25A,$$
$$d_8^{12} = d_8^{21} = 5P[40P(F + 2n) - 5E - 3],$$
$$d_8^{22} = 5N[40(D + 2nN) - 3] - 25C,$$
$$d_9^{11} = M(4B + 4p + 9A),$$
$$d_9^{12} = d_9^{21} = P^2(4F - 4p + 9E),$$
$$d_9^{22} = N(4D + 4p + 9C),$$

where

$$M = \sqrt{\frac{1+(f_y)^2}{1+f_x^2+f_y^2}}, \quad N = \sqrt{\frac{1+(f_x)^2}{1+f_x^2+f_y^2}}, \quad P = \sqrt{-\frac{f_x f_y}{1+f_x^2+f_y^2}},$$

and

$$A = \sqrt{R_1 - R_2 + 2M^2\theta_5}, \qquad B = \sqrt{R_1 - R_2 + M^2\theta_6},$$
$$C = \sqrt{R_3 - R_4 + 2N^2\theta_5}, \qquad D = \sqrt{R_3 - R_4 + N^2\theta_6},$$

and

$$E = \sqrt{R_5}, \quad F = \sqrt{R_5 + \frac{4}{c}},$$

where

$$R_1 = 2\sqrt[3]{2\frac{f_x^4(1+f_y^2)}{(1+f_x^2+f_y^2)^5} + 8\varepsilon_1\frac{f_x^2(1+f_y^2)^2}{(1+f_x^2+f_y^2)^4} + 8\varepsilon_4\frac{(1+f_y^2)^3}{(1+f_x^2+f_y^2)^3}},$$
$$R_2 = 4\sqrt[3]{2\varepsilon_2\frac{(1+f_y^2)^3}{(1+f_x^2+f_y^2)^3} + \theta_4^2\frac{f_x^2(1+f_y^2)^2}{(1+f_x^2+f_y^2)^4}},$$
$$R_3 = 2\sqrt[3]{2\frac{f_y^4(1+f_x^2)}{(1+f_x^2+f_y^2)^5} + 8\varepsilon_1\frac{f_y^2(1+f_x^2)^2}{(1+f_x^2+f_y^2)^4} + 8\varepsilon_4\frac{(1+f_x^2)^3}{(1+f_x^2+f_y^2)^3}},$$

$$R_4 = 4\sqrt[3]{2\varepsilon_2 \frac{(1+f_x^2)^3}{(1+f_x^2+f_y^2)^3} + \theta_4^2 \frac{f_y^2(1+f_x^2)^2}{(1+f_x^2+f_y^2)^4}},$$

$$R_5 = 2\left(\sqrt[3]{2(\frac{1}{c} - 2\varepsilon_1)^2 + 8\varepsilon_2} + 2\theta_5 - 2\sqrt[3]{2\varepsilon_2 - \frac{2}{c}\theta_4^2}\right),$$

and

$$c = 1 + f_x^2 + f_y^2,$$

$$m = \frac{1}{1+f_x^2+f_y^2},$$

$$n = -\frac{29 + 9f_x^2 + 9f_y^2}{29(1+f_x^2+f_y^2)},$$

$$p = \frac{1 - f_x^2 - f_y^2}{2(1+f_x^2+f_y^2)},$$

$$\theta_1 = -\frac{258c^3 - 1256c^2 + 1684c - 712}{12^3c^3},$$

$$\theta_2 = \frac{81c^2 + 90c + 48}{12^2c^2},$$

$$\theta_3 = -\left(\frac{9c-8}{12c}\right)^2,$$

$$\theta_4 = \frac{3c-2}{6c},$$

$$\theta_5 = \frac{12c-11}{12c},$$

$$\theta_6 = \frac{12c^2 + 13c - 24}{6c^2},$$

$$\varepsilon_1 = \frac{-45c^2 - 138c - 32}{12^2c^2},$$

$$\varepsilon_2 = \frac{-2187c^4 + 41796c^3 - 15660c^2 + 24768c - 768}{12^4c^4},$$

$$\varepsilon_3 = \frac{921c^4 + 14732c^3 - 1084c^2 + 6832c - 256}{12^3c^4},$$

$$\varepsilon_4 = \frac{13077c^4 + 189204c^3 + 8916c^2 + 90816c - 2048}{12^4c^4},$$

getting in this way all the four quadric form which allow us to find, in T^*M, using (4.1), the $\mathcal{L}$-dual of our particular Matsumoto space from above.

For the above construction, we need to analyze the existence of the expressions under the radicals. M, N allways exist.

First of all, because of the radical in the expression of P we must have $f_x f_y \leq 0$. If $f_x f_y = 0$ we get $d_2^{12} = d_2^{21} = 0$ and $d_4^{12} = d_4^{21} = 0$, $d_8^{12} = d_8^{21} = 0$, $d_9^{12} = d_9^{21} = 0$.

Let us put $\Delta = (\varepsilon_1 - \theta_4^2)^2 - 4(\varepsilon_4 - 2\varepsilon_2)$ and $S = 4(\varepsilon_4 - 2\varepsilon_2)$. Therefore, we have:

If $\Delta < 0$ then $R_1 - R_2 \geq 0$ and $R_3 - R_4 \geq 0$ for any value of c. This allows us to conclude that A, B, C, D always exist.

If $\Delta \geq 0$ and $c \in \left[1, \frac{4}{3}\right]$ or $\Delta \geq 0$ and $S \geq 0$, then $R_1 - R_2 \geq 0$ and $R_3 - R_4 \geq 0$ proving the existence of A, B, C, D.

We also need to have $R_5 \geq 0$. But this depends on the value of $c \geq 1$. For example, if $c \in \left[1, \frac{4}{3}\right]$ we have $R_5 \in [-0,0701; 2,1898]$.

To complete our discussion, we mention here the following result [SS1]: if $f_x^2 + f_y^2 \leq \frac{1}{3}$ i.e. $1 \leq c \leq \frac{4}{3}$, then $\frac{f_x^2 + f_y^2}{1 + f_x^2 + f_y^2} \leq \frac{1}{4}$ and the fundamental tensor g^{ij} of Matsomoto space $F = \frac{\alpha^2}{\alpha - \beta}$ with α and β defined above is positively defined, or equivalently, the indicatrix is convex.

Example 2. Let us consider the surface S to be a plane, $z = f(x, y) = \frac{1}{2}x$.

The convexity condition for the indicatrix is satisfied, i.e.: $f_x^2 + f_y^2 = \frac{1}{4} < \frac{1}{3}$. Now, $f_x = \frac{1}{2}$, $f_y = 0$,

$$(a_{ij}) = \begin{pmatrix} \dfrac{5}{4} & 0 \\ 0 & 1 \end{pmatrix}, \quad \det \|a_{ij}\| = \frac{5}{4}, \quad (a^{ij}) = \begin{pmatrix} \dfrac{4}{5} & 0 \\ 0 & 1 \end{pmatrix},$$

and $\tilde{b}^1 = \frac{2}{5}$, $\tilde{b}^2 = 0$ and $b^2 = \frac{1}{5}$.

Following the calculus from above, we get:

$$d_2^{11} = 10.7621695,$$
$$d_2^{12} = d_2^{21} = 0,$$
$$d_2^{22} = 18.5916118,$$
$$d_4^{11} = 4.1619406,$$
$$d_4^{12} = d_4^{21} = 0,$$
$$d_4^{22} = 3.3692342,$$
$$d_8^{11} = 255.0575035,$$

$$d_8^{12} = d_8^{21} = 0,$$

$$d_8^{22} = 185.6868118,$$

$$d_9^{11} = 24.6023378,$$

$$d_9^{12} = d_9^{21} = 0,$$

$$d_9^{22} = 23.1147203,$$

and for the four quadratic forms and β^* we get:

$$\alpha_2^{*2} = 10.7621695t^2 + 18.5916118s^2,$$

$$\alpha_4^{*2} = 4.1619406t^2 + 3.3692342s^2,$$

$$\alpha_8^{*2} = 255.0575035t^2 + 185.6868118s^2,$$

$$\alpha_9^{*2} = 24.6023378t^2 + 23.1147203s^2,$$

$$\beta^* = 0.4t.$$

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