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An existence result for a parametric variational inequality

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Abstract. In this paper we study a variational inequality formulated using the supremum over a set of parameters.

1. Introduction

Existence of solutions for variational inequalities generated by nonlinear mappings was studied a lot in the last 40 years, especially after the appearance of the paper [6] (see also [5]). Among them, an important place is occupied by those that occur, for instance, in the study of (necessary) optimality conditions for variational problems with obstacles.

If we consider a variational problem of the form

$$\max_{\lambda\in\Lambda}\int_{\Omega} L(\lambda, t, u(t), \nabla(t))dt \to \min!, \quad u \in K$$

then the necessary condition for the optimality usually appears (by some natural conditions) as a nonlinear variational inequality governed by a set -valued map (see for instance [7]). Our aim in this paper is to study variational inequalities of this kind. A related problem was studied, in a different framework, in [4].

The paper is organized as follows. In Section 2 we give the exact formulation of the problem and some auxiliary results. The main result of the paper (proved in Section 3, Theorem 8) is an existence theorem for nonlinear variational inequalities governed by a parameter depending set-valued mapping. In the particular case

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when the set of parameters is a singleton, our result reduces to a classical one (see [6] or [8], p. 42).

2. Setting of the problem

Let $\Omega \subset \mathbb{R}^n$ be open, bounded, nonempty; let K be a nonempty, convex and closed subset of the Sobolev space $H_0^1(\Omega)$.
Consider the problem:

$$u \in K \text{ such that } \sup_{p\in P(u)} \left\{ \int_\Omega A(p,x,u(x),\nabla u(x))\nabla(w-u)(x)dx \right.$$
$$\left. + \int_\Omega a(p,x,u(x),\nabla u(x))(w-u)(x)dx \right\} \geq 0, \quad \forall\, w \in K \tag{1}$$

where $P : H_0^1(\Omega) \to 2^\Lambda$ is a set-valued mapping, with Λ a metric space and $P(u) \neq \emptyset$; $A = (a_1,\ldots,a_n) : \Lambda\times\Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$, $a_0 : \Lambda\times\Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ are functions in the following hypotheses:

(H1) $a_j(p,x,\eta,\xi)$ is measurable in the second variable x and continuous in (p,η,ξ), for $j = 0,\ldots,n$;

(H2) $|a_j(p,x,\eta,\xi)| \leq c(k(x)+|\eta|+\|\xi\|)$, for a.e. $x\in\Omega$, for each $\eta\in\mathbb{R}$, $\xi\in\mathbb{R}^n$, $p\in\Lambda$, with c a positive constant and $k\in L^2(\Omega)$, with positive values, for $j=0,\ldots,n$;

(H3) $\sum_{j=1}^n (a_j(p,x,\eta,\xi)-a_j(p,x,\eta,\tilde{\xi}))(\xi_j-\tilde{\xi}_j) > 0$, for a.e. $x\in\Omega$, for each $\eta\in\mathbb{R}$, $\xi\neq\tilde{\xi}\in\mathbb{R}^n$, $p\in\Lambda$;

(H4) $\sum_{j=1}^n a_j(p,x,\eta,\xi)\xi_j \geq c_1\|\xi\|^2 - c_2$, for a.e. $x\in\Omega$, for each $\eta\in\mathbb{R}$, $\xi\in\mathbb{R}^n$, $p\in\Lambda$, with c_1, c_2 positive constants.

We define, similar to [8], the operator $\mathcal{A} : \Lambda\times H_0^1(\Omega) \to (H_0^1(\Omega))^*$, by

$$\langle w, \mathcal{A}(p,u)\rangle = \int_\Omega A(p,x,u(x),\nabla u(x))\nabla w(x)dx$$
$$+ \int_\Omega a_0(p,x,u(x),\nabla u(x))w(x)dx, \tag{2}$$

for each $w\in H_0^1(\Omega)$. Then the variational inequality (1) can be written:

$$u\in K \text{ such that } \sup_{p\in P(u)} \langle w-u, \mathcal{A}(p,u)\rangle, \quad \forall\, w\in K.$$

We make also the hypotheses:

(H5) For any $u \in K$, fixed, the set

$$C(u) \stackrel{not}{=} \{\mathcal{A}(p,u) \mid \; p \in P(u)\} \subset (H_0^1(\Omega))^*$$

is convex. (this is true, for instance, if the functions a_j are linear with respect to p and $P(u)$ is convex).

(H6) $P(u) \subset \Lambda$ is compact, for each $u \in K$.

(H7) The set-valued function $P : H_0^1(\Omega) \to 2^\Lambda$ is upper semi-continuous (with the weak topology in $H_0^1(\Omega)$),

(H8) There exists $B \subset H_0^1(\Omega)$, weakly compact and $v_0 \in K$ such that

$$\sup_{p\in P(u)} \langle v_0 - u, \mathcal{A}(p,u)\rangle < 0,$$

for every $u \in K \setminus B$.

(H9) For $p_n \to p$, $u_n \rightharpoonup u$, $p_n \in P(u_n)$, $w \in H_0^1(\Omega)$ fixed, if

$$\sup_{p\in P(u_n)} \langle tw + (1-t)u - u_n, \mathcal{A}(p,u_n)\rangle \geq 0, \quad \forall\, t \in [0,1]$$

then

$$\limsup_n \{\langle tw + (1-t)u - u_n, \mathcal{A}(p_n,u_n) - \mathcal{A}(p,u_n)\rangle\} \leq 0, \quad \forall\, t \in [0,1].$$

(H10) The functions a_j, $j = 0,\ldots,n$ are Lipschitz with respect to p, that is,

$$|a_j(p_1,x,\eta,\xi) - a_j(p_2,x,\eta,\xi)| \leq \theta_j(x)\mathrm{d}(p_1,p_2),$$

with $\theta_j \in L^2(\Omega)$ some functions with positive values.

Theorem 1 (Leray–Lions, [8], p. 76)**.** *In the hypotheses* (H1)–(H4)*, for any fixed parameter $p \in \Lambda$, the operator $\mathcal{A}(p,\cdot) : H_0^1(\Omega) \to (H_0^1(\Omega))^*$ is pseudo-monotone (in the sense of* BRÉZIS, [1]*), that is, for each sequence $\{u_n\} \subset H_0^1(\Omega)$, $u_n \rightharpoonup u$ and*

$$\limsup_n \langle u_n - u, \mathcal{A}(p,u_n)\rangle \leq 0$$

imply

$$\liminf_n \langle u_n - w, \mathcal{A}(p,u_n)\rangle \geq \langle u - w, \mathcal{A}(p,u)\rangle,$$

for each $w \in H_0^1(\Omega)$.

Lemma 2. *In the hypotheses* (H1) *and* (H2) *the set $C(u)$ is bounded.*

Proof. Let $\mathcal{A}(p,u) \in C(u)$. Using hypothesis (H2), Hölder's and Poincare's inequalities, we have

$$\begin{aligned}
\|\mathcal{A}(p,u)\|_{(H_0^1(\Omega))^*} &= \sup_{\|w\|_{H_0^1(\Omega)} \le 1} |\langle w, \mathcal{A}(p,u)\rangle| \\
&= \sup_{\|w\|_{H_0^1(\Omega)} \le 1} \Big\{ \Big| \int_\Omega A(p,x,u(x),\nabla u(x)) \nabla w(x) dx \\
&\quad + \int_\Omega a_0(p,x,u(x),\nabla u(x)) w(x) dx \Big| \Big\} \\
&\le \sup_{\|w\|_{H_0^1(\Omega)} \le 1} 2c \Big(\int_\Omega |k(x)|^2 dx \Big)^{1/2} \|w\|_{H_0^1(\Omega)} \\
&\quad + 4c \|w\|_{H_0^1(\Omega)} \|u\|_{H_0^1(\Omega)} \le 2c\|k\|_{L^2(\Omega)} + 4c\|u\|_{H_0^1(\Omega)},
\end{aligned}$$

so the set $C(u)$ is bounded. □

Lemma 3. *If* (H1), (H2), (H6) *and* (H10) *are satisfied, then the set* $C(u)$ *is weakly* closed.*

Proof. It follows directly from (H6) and the continuity of the mapping $p \mapsto \langle w, \mathcal{A}(p,u)\rangle$ (H10). In fact, consider a weak* convergent sequence $\{v_n^*\} \subset C(u)$ such that $v_n^* \xrightarrow{*} v^*$. From $v_n^* \in C(u)$ it follows that there exists $p_n \in P(u)$ such that

$$\langle w, v_n^* \rangle = \langle w, \mathcal{A}(p,u)\rangle, \quad \forall\, w \in H_0^1(\Omega). \tag{3}$$

Hypothesis (H6) implies the existence of a subsequence $p_{n_k} \in P(u)$ with the property $p_{n_k} \to p \in P(u)$. Considering equation (3) for this subsequence and passing to the limit we get $\langle w, v^* \rangle = \langle w, \mathcal{A}(p,u)\rangle$, for each $w \in H_0^1(\Omega)$, that is $v^* \in C(u)$. □

We use the following Lemma:

Lemma 4 ([4]). *Let E be a real Hilbert space, $x, y \in E$, $C \subset E^*$, convex, bounded, closed. If*

$$\sup_{z^* \in C} \langle (1-t)x + ty, z^* \rangle \ge 0, \quad \forall\, t \in [0,1],$$

then there exists $z^* \in C$ such that $\langle (1-t)x + ty, z^* \rangle \ge 0$, for every $t \in [0,1]$.

Applying this in our situation we get:

Lemma 5. *In the hypotheses* (H1)–(H6) *and* (H10), *if* $v, w \in H_0^1(\Omega)$ *and*

$$\sup_{p\in P(u)} \langle (1-t)v + tw, \mathcal{A}(p,u)\rangle \geq 0, \quad \forall\, t \in [0,1],$$

then there exists $\bar{p} \in P(u)$ *such that*

$$\langle (1-t)v + tw, \mathcal{A}(\bar{p},u)\rangle \geq 0, \quad \forall\, t \in [0,1].$$

Proof. We apply Lemma 4 for $E = H_0^1(\Omega)$, $v, w \in H_0^1(\Omega)$ and $C = C(u) \in (H_0^1(\Omega))^*$.

Using hypothesis (H5), Lemma 2 and Lemma 3, we get that $C(u)$ is convex, bounded and weakly* closed. We have

$$\sup_{z^*\in C(u)} \langle (1-t)v + tw, z^*\rangle = \sup_{p\in P(u)} \langle (1-t)v + tw, \mathcal{A}(p,u)\rangle,$$

for every $t \in [0,1]$. From Lemma 4, there exists an element $\bar{z} \in C(u)$ such that $\langle (1-t)v + tw, \bar{z}\rangle \geq 0$, for every $t \in [0,1]$, that is there exists $\bar{p} \in P(u)$ such that $\langle (1-t)v + tw, \mathcal{A}(\bar{p},u)\rangle \geq 0$, for every $t \in [0,1]$. $\square$

Lemma 6 ([3])**.** *Let U and V be topological spaces, $G : U \to 2^V$ a set-valued mapping and $g : U \times V \to \mathbb{R}$. Denote by $h : U \to \mathbb{R}$, $h(u) = \sup_{v\in G(u)} g(u,v)$ the marginal function. If the conditions:*

(i) *g is upper semi-continuous on $U \times V$,*

(ii) *$G(u_0)$ is compact for some $u_0 \in U$,*

(iii) *G is upper semi-continuous at u_0,*

are satisfied, then h is upper semi-continuous at u_0.

3. The main result

To prove the existence of a solution to the problem (1) we will use a generalization of the Ky Fan Intersection Lemma:

Lemma 7 ([2])**.** *Let V be a topological vector space, $H \subset V$ and $F : H \to 2^V$ such that:*

(i) $\operatorname{cl} F(x_0)$ *is compact for some $x_0 \in H$,*

(ii) *for every $x_1, \dots, x_n \in H$, $co\{x_1, \dots, x_n\} \subset \bigcup_{i=1}^n F(x_i)$,*

(iii) *for each $x \in H$, the intersection of $F(x)$ with any finite dimensional subspace of V is closed,*

(iv) *for every line segment D of V,*

$$\mathrm{cl}\Big(\bigcap_{x\in H\cap D} F(x)\Big)\cap D = \Big(\bigcap_{x\in H\cap D} F(x)\Big)\cap D,$$

Then $\bigcap_{x\in E} F(x)\neq\emptyset$.

If H is convex, closed and $F(x)\subset H$ for every $x\in H$, then the hypothesis (iv) can be replaced with: (iv') for every line segment D of H,

$$\mathrm{cl}\Big(\bigcap_{x\in D} F(x)\Big)\cap D = \Big(\bigcap_{x\in D} F(x)\Big)\cap D.$$

We can prove now:

Theorem 8. *In the hypotheses* (H1)–(H10), *the variational inequality* (1) *has at least a solution.*

Proof. For each $v\in K$, denote $F(v)=\{u\in K \mid \sup_{p\in P(u)}\langle v-u,\mathcal{A}(p,u)\rangle\geq 0\}$. It is obvious that, if $u\in\bigcap_{v\in K} F(v)$, then u is a solution of (1).

With $V=H_0^1(\Omega)$ and $H=K$, we check the four conditions of Lemma 7.

(i) Let $v_0\in K$ be the element mentioned in hypothesis (H8). We have that $F(v_0)\subset B$ (if not, assuming that there exists $u\in F(v_0)$ and $u\notin B$, then $\sup_{p\in P(u)}\langle v_0-u,\mathcal{A}(p,u)\rangle<0$ is a contradiction.) Then w-cl$F(v_0)\subset B$, and since B is weakly compact, w-cl$F(v_0)$ is also weakly compact.

(ii) Let $u_1,\dots,u_n\in K$. Suppose that there exist $\alpha_1,\dots,\alpha_n\in[0,1]$ with $\sum_{j=1}^n\alpha_j=1$ such that $\sum_{j=1}^n\alpha_j u_j\notin\bigcup_{j=1}^n F(u_j)$, which means that, for each $j=1,\dots,n$, $\bar u=\sum_{j=1}^n\alpha_j u_j\notin F(u_j)$ and so

$$\sup_{p\in P(\bar u)}\langle u_j-\bar u,\mathcal{A}(p,\bar u)\rangle<0,\quad \forall\ j=1,\dots,n.$$

We fix $p\in P(\bar u)$ and get

$$\begin{aligned}0=\langle \bar u-\bar u,\mathcal{A}(p,\bar u)\rangle &= \left\langle \sum_{j=1}^n\alpha_j u_j-\Big(\sum_{j=1}^n\alpha_j\Big)\bar u,\mathcal{A}(p,\bar u)\right\rangle\\ &=\left\langle\sum_{j=1}^n\alpha_j(u_j-\bar u),\mathcal{A}(p,\bar u)\right\rangle=\sum_{j=1}^n\alpha_j\langle u_j-\bar u,\mathcal{A}(p,\bar u)\rangle<0,\end{aligned}$$

which is a contradiction.

(iii) Let $Z \subset H_0^1(\Omega)$ be a finite dimensional subspace and let $v \in K$, fixed. We want to prove that $F(v) \cap Z$ is a closed set.

Consider a sequence $\{u_n\} \subset F(v) \cap Z$, with $u_n \to u$. Then $u_n \in K \cap Z$ and so $u \in K \cap Z$.

Denote $g : H_0^1(\Omega) \times \Lambda \to \mathbb{R}$, $g(u,p) = \langle v - u, \mathcal{A}(p,u)\rangle$. From (H7) we have that P is upper semi-continuous also with the strong topology on $H_0^1(\Omega)$. Using this, (H6) and the continuity of the operator $\mathcal{A}$ (see [8], p. 74), the hypotheses of Lemma 6 are satisfied and we get that the mapping $u \mapsto \sup_{p\in P(u)}\langle v-u, \mathcal{A}(p,u)\rangle$ is upper semi continuous, which means that,

$$0 \le \limsup_n \sup_{p\in P(u_n)} \langle v - u_n, \mathcal{A}(p,u_n)\rangle \le \sup_{p\in P(u)} \langle v - u, \mathcal{A}(p,u)\rangle,$$

and so $u \in F(v)$.

(iv') We prove first that, for each $w \in K$,

$$\text{from } u \in \text{w-cl}\Big(\bigcap_{v\in[u,w]} F(v)\Big) \text{ it follows that } \quad u \in \Big(\bigcap_{v\in[u,w]} F(v)\Big). \tag{4}$$

Let $\{u_n\} \subset \bigcap_{v\in[u,w]} F(v)$ be a sequence such that $u_n \rightharpoonup u$. We have $u_n \in F(v)$, for each $v \in [u,w]$, that is, $u_n \in K$ and

$$\sup_{p\in P(u_n)} \langle tw + (1-t)u - u_n, \mathcal{A}(p,u_n)\rangle \ge 0, \quad \forall\, t \in [0,1].$$

The set K is convex and closed, so it is also weakly closed and $u \in K$.

From Lemma 5 we have that for each n there exists $p_n \in P(u_n)$ such that

$$\langle tw + (1-t)u - u_n, \mathcal{A}(p_n,u_n)\rangle \ge 0, \quad \forall\, t \in [0,1]. \tag{5}$$

For $t = 0$, we get $\langle u - u_n, \mathcal{A}(p_n,u_n)\rangle \ge 0$, that is $\langle u_n - u, \mathcal{A}(p_n,u_n)\rangle \le 0$ and

$$\limsup_n \langle u_n - u, \mathcal{A}(p_n,u_n)\rangle \le 0. \tag{6}$$

Since P has compact values and is upper semi-continuous, it follows (according to [3], vol. I, p. 41) that there exists a subsequence, still denoted by p_n such that $p_n \in P(u_n)$ and $p_n \to p$, $p \in P(u)$. For this subsequence, (6) holds. We have:

$$\begin{aligned}\limsup_n \langle u_n - u, \mathcal{A}(p,u_n)\rangle &\le \limsup_n \{\langle u_n - u, \mathcal{A}(p,u_n)\rangle - \langle u_n - u, \mathcal{A}(p_n,u_n)\rangle\} \\ &\quad + \limsup_n \langle u_n - u, \mathcal{A}(p_n,u_n)\rangle \le 0,\end{aligned}$$

from (6) and (H9) with $t := 0$. According to Lemma 1, for p fixed, the operator $\mathcal{A}$ is B-pseudomonotone, so

$$\liminf_n \langle u_n - w, \mathcal{A}(p, u_n)\rangle \geq \langle u - w, \mathcal{A}(p, u)\rangle \tag{7}$$

Next, using (7) and (H9) (with $t := 1$),

$$\begin{aligned}
\liminf_n \langle u_n - w, \mathcal{A}(p_n, u_n)\rangle &\geq \liminf_n \{\langle u_n - w, \mathcal{A}(p_n, u_n)\rangle - \langle u_n - w, \mathcal{A}(p, u_n)\rangle\} \\
&\quad + \liminf_n \langle u_n - w, \mathcal{A}(p, u_n)\rangle \\
&= -\limsup_n \{\langle u_n - w, \mathcal{A}(p, u_n)\rangle - \langle u_n - w, \mathcal{A}(p_n, u_n)\rangle\} \\
&\quad + \liminf_n \langle u_n - w, \mathcal{A}(p, u_n)\rangle \geq \langle u - w, \mathcal{A}(p, u)\rangle.
\end{aligned}$$

It follows, using (5) with $t = 1$, that

$$\langle w - u, \mathcal{A}(p, u)\rangle \geq -\liminf_n \langle u_n - w, \mathcal{A}(p_n, u_n)\rangle = \limsup_n \langle w - u_n, \mathcal{A}(p_n, u_n)\rangle \geq 0$$

so $\sup_{p\in P(u)} \langle w - u, \mathcal{A}(p, u)\rangle \geq 0$. For $t \in [0, 1]$, we have

$$\sup_{p\in P(u)} \langle tw + (1 - t)u - u, \mathcal{A}(p, u)\rangle = t \sup_{p\in P(u)} \langle w - u, \mathcal{A}(p, u)\rangle \geq 0,$$

which means that, for each $t \in [0, 1]$, $u \in F(tw + (1 - t)u)$, so

$$u \in \bigcap_{v\in[u,w]} F(v),$$

which proves (4). Consider now $z \in \text{w-cl}\big(\bigcap_{v\in[u,w]} F(v)\big) \cap [u, w]$. From here

$$z \in \text{w-cl}\Bigg(\bigcap_{v\in[u,z]} F(v)\Bigg) \cap [u, z] \quad \text{and} \quad z \in \text{w-cl}\Bigg(\bigcap_{v\in[z,w]} F(v)\Bigg) \cap [z, w].$$

According to (4), it follows that

$$z \in \Bigg(\bigcap_{v\in[u,z]} F(v)\Bigg) \cap [u, z] \quad \text{and} \quad z \in \Bigg(\bigcap_{v\in[z,w]} F(v)\Bigg) \cap [z, w].$$

and next $z \in \big(\bigcap_{v\in[u,w]} F(v)\big) \cap [u, w]$, which concludes the proof. $\square$

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