

## Some limit theorems via Lévy distance

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*Dedicated to the 100<sup>th</sup> anniversary of the birthday of Béla Györfi*

**Abstract.** In this note, we present a unified approach for the proofs of two results of probability theory, namely the (so-called) *global version of the central limit theorem* and the *complete convergence theorem*. Key tool is an appropriate upper bound for the deviation of two distribution functions in terms of the Lévy distance.

### 1. Introduction

Let  $F$  and  $G$  be two distribution functions on  $\mathbf{R}$  and denote by  $L = \mathcal{L}(F, G)$  the Lévy distance between  $F$  and  $G$ , that is,

$$L = \inf \mathbb{H}, \quad (1.1)$$

where  $\mathbb{H} = \mathbb{H}(F, G) = \{h : G(x - h) - h \leq F(x) \leq G(x + h) + h \text{ for all } x \in \mathbf{R}\}$ . On using the notation  $\Phi$  for the standard normal distribution function, the weak convergence of a sequence of distribution functions  $\{F_n\}$  to  $\Phi$  will be denoted by

$$F_n \xrightarrow{w} \Phi, \quad n \rightarrow \infty.$$

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By definition, weak convergence of a sequence of distribution functions  $\{F_n\}$  to a distribution function  $G$  means that

$$\lim_{n \rightarrow \infty} F_n(x) = G(x), \quad x \in \mathcal{C}(G), \quad (1.2)$$

where  $\mathcal{C}(G)$  stands for the set of continuity points of  $G$ . Since  $\Phi$  is everywhere continuous, weak convergence to  $\Phi$  coincides with the pointwise convergence, even with the uniform convergence on  $\mathbf{R}$ .

It is also known that weak convergence is equivalent to the convergence to zero of the Lévy distances  $\mathcal{L}(F_n, G)$  (see, for example, GNEDENKO and KOLMOGOROV [9]).

We also denote by  $\Delta = \Delta(F, G)$  the uniform distance between distribution functions  $F$  and  $G$ , namely

$$\Delta = \Delta(F, G) = \sup_{x \in \mathbf{R}} |F(x) - G(x)|. \quad (1.3)$$

Since  $\Delta(F, G) \in \mathbb{H}(F, G)$ , we have  $L \leq \Delta$ .

The weak convergence of a sequence of distribution functions  $\{F_n\}$  to a *continuous* distribution function  $G$  is equivalent to  $\Delta(F_n, G) \rightarrow 0$  (see, for example, GNEDENKO and KOLMOGOROV [9]).

Results about the approximation of a sequence of distribution functions  $\{F_n\}$  by the normal law are called *central limit theorems* in probability theory. The classical case deals with a sequence of distribution functions  $F_n$  corresponding to centered and normalized partial sums of independent random variables. Approximations in terms of the uniform distance have a long history. Confer, e.g., ESSEEN [7] for the roots of the modern theory. ESSEEN's [7] result has been generalized in KOLODYAZHNYĬ [12].

**Theorem 1.1** (KOLODYAZHNYĬ [12]). *Let  $p > 0$  and*

$$\int_{-\infty}^{\infty} |x|^p dF(x) < \infty.$$

*Let  $\Delta$  be the uniform distance between  $F$  and  $\Phi$  and assume that  $0 < L \leq e^{-1/2}$ . Then there exists a universal constant  $c_\Delta$ , depending only on  $p$ , such that*

$$|F(x) - \Phi(x)| \leq \frac{\lambda_p + c_\Delta \Delta (\ln \frac{1}{\Delta})^{p/2}}{1 + |x|^p} \quad (1.4)$$

*for all  $x \in \mathbf{R}$ , where*

$$\lambda_p = \left| \int_{-\infty}^{\infty} |x|^p dF(x) - \int_{-\infty}^{\infty} |x|^p d\Phi(x) \right|. \quad (1.5)$$

The following upper estimate for the pointwise deviation between a distribution function  $F$  and the standard normal one  $\Phi$  has been given in INDLEKOFER and KLESOV [11]. Note that  $F$  in Theorem 1.2 below does not necessarily correspond to a sum of independent random variables.

**Theorem 1.2** (INDLEKOFER and KLESOV [11]). *Let  $p > 0$  and*

$$\int_{-\infty}^{\infty} |x|^p dF(x) < \infty.$$

*Let  $L$  be the Lévy distance between  $F$  and  $\Phi$  and assume that  $0 < L \leq e^{-1/2}$ . Then there exists a universal constant  $c_L$ , depending only on  $p$ , such that*

$$|F(x) - \Phi(x)| \leq \frac{\lambda_p + c_L L (\ln \frac{1}{L})^{p/2}}{1 + |x|^p} \quad (1.6)$$

for all  $x \in \mathbf{R}$ , where  $\lambda_p$  is defined by (1.5).

*Remark 1.1.* There are some cases where one of Theorems 1.1 and 1.2 implies the other one. The function  $x (\log \frac{1}{x})^{p/2}$  increases in the interval  $(0, e^{-p/2})$  and decreases in  $(e^{-p/2}, 1)$ . Thus (1.6) implies (1.4) if

$$\Delta \leq \min \left\{ \frac{1}{\sqrt{e^p}}, \frac{1}{\sqrt{e}} \right\}. \quad (1.7)$$

Moreover,  $c_L = c_\Delta$  in this case.

On the other hand, (1.4) implies (1.6) if  $p \geq 1$  and

$$\frac{1}{\sqrt{e^p}} \leq L \leq \Delta \frac{1}{\sqrt{e}}. \quad (1.8)$$

Again,  $c_L = c_\Delta$  in this case. The case (1.7) fits better the context of the central limit theorem as compared to the case of (1.8).

Since

$$\Delta \leq \left( 1 + \frac{1}{\sqrt{2\pi}} \right) L$$

if  $G = \Phi$ , there are several other interrelations between Theorems 1.1 and 1.2. In what follows we use (1.6) rather than (1.4). This can be explained by the fact that other functions  $G$  (even discontinuous) can be used in (1.6) in place of  $\Phi$ , while (1.4) has no nice generalizations in the discontinuous case. Theorem 1.2 for other  $G$  instead of  $\Phi$  will be discussed elsewhere.

The simplest case of Theorem 1.2 corresponds to  $p = 2$ .

**Corollary 1.1.** *Let a distribution function  $F$  satisfy the condition*

$$\int_{-\infty}^{\infty} x^2 dF(x) = 1.$$

*If the Lévy distance  $L = \mathcal{L}(F, \Phi)$  is such that  $0 < L \leq e^{-1/2}$ , then*

$$|F(x) - \Phi(x)| \leq \frac{AL \ln \frac{1}{L}}{1 + x^2} \quad (1.9)$$

*with some universal constant  $A > 0$  and for all  $x \in \mathbf{R}$ .*

The main aim of this note is to demonstrate that Theorem 1.2 can be applied in various situations, among which are the (so-called) *global versions of the central limit theorem* due to AGNEW [2] (see also AGNEW [3]–[4] for further extensions) and the *complete convergence theorem* due to HSU and ROBBINS [10], for which an extensive literature exists (cf., e.g., BAUM and KATZ [5] and SPITZER [19], just to mention a few).

## 2. Global version of the central limit theorem

AGNEW [2] has apparently been the first to study the relationship between weak convergence and the convergence to zero of the integral in (2.2) below. We should mention, however, that in Agnew's setting the limit distribution function can be arbitrary and not just  $\Phi$ . Nevertheless, we restrict our considerations here to  $\Phi$  as a weak limit in order to highlight the main features of our approach. Moreover, for the sake of comparison, we state the results of other authors also with  $\Phi$ , even if they have been proved for more general limiting distribution functions. For example, from AGNEW [2] we have the following result.

**Theorem 2.1** (AGNEW [2]). *Let  $\{F_n\}$  be a sequence of distribution functions such that*

$$\int_{-\infty}^{\infty} x dF_n(x) = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} x^2 dF_n(x) = 1. \quad (2.1)$$

*Assume that*

$$F_n \xrightarrow{w} \Phi, \quad n \rightarrow \infty.$$

*Then, for all  $r > \frac{1}{2}$ ,*

$$\int_{-\infty}^{\infty} |F_n(x) - \Phi(x)|^r dx \rightarrow 0, \quad n \rightarrow \infty. \quad (2.2)$$

Similar results can be obtained via Theorem 1.1, too (see, [16] and [17]). Other methods of proof are due to STUDNEV and IGNAT [20] and KRUGLOV [13].

In fact, AGNEW [2] proved that the weak convergence is equivalent to (2.2) in a certain sense. Note that the distribution functions  $\{F_n\}$  in Theorem 2.1 are arbitrary and need not correspond to partial sums of independent random variables. The latter more specific and more typical case will be considered next.

**Corollary 2.1.** *Let  $\{X_n\}$  be a sequence of independent, identically distributed random variables and let  $\{S_n\}$  denote their partial sums. Assume that*

$$\mathbf{E}X_1 = 0 \quad \text{and} \quad \mathbf{E}X_1^2 = 1.$$

*If  $r > \frac{1}{2}$ , then (2.2) holds, where  $F_n$  denotes the distribution function of  $n^{-1/2}S_n$ .*

This result was generalized by ESSEEN [8] to the case of independent, but not necessarily identically distributed random variables  $\{X_n\}$ , provided they satisfy the central limit theorem. A further extension is due to de ACOSTA and GINÉ [1].

Of course, the smaller the value of  $r > 0$ , the stronger is the convergence in (2.2). So, it is natural to ask whether or not the restriction  $r > \frac{1}{2}$  in the above results can be weakened to  $r > 0$ . A first positive answer has been given by NISHIMURA [15]. However, as pointed out in ROSALSKY [18], the argument in [15] is incomplete and the conditions can even be improved.

**Theorem 2.2** (LAUBE [14] and ROSALSKY [18]). *Let  $\{F_n\}$  be a sequence of distribution functions such that*

$$\sup_{n \geq 1} \int_{-\infty}^{\infty} |x|^p dF_n(x) < \infty \quad (2.3)$$

*for some  $p > 1$ . Assume that  $\{F_n\}$  converges weakly to  $\Phi$  as  $n \rightarrow \infty$ . Then (2.2) holds for all  $r > \frac{1}{p}$ .*

In view of Theorem 2.2 it is natural to conjecture that (2.2) of Theorem 2.1 holds for all  $r > 0$ , provided the moments of any order exist. This is indeed the case, as was mentioned by de ACOSTA and GINÉ [1].

Just for the sake of demonstration, we provide different proofs of Theorems 2.1 and 2.2 via estimates for the Lévy distance. A similar method via estimates for the uniform distance is used in [16] and [17] for some particular cases. Different methods are presented in [13] and [20].

**PROOF OF THEOREM 2.1.** Set  $L_n = \mathcal{L}(F_n, \Phi)$ . We assume without loss of generality (w.l.o.g.) that  $L_n < e^{-1/2}$  for all  $n \geq 1$ . Then, by Corollary 1.1,

$$|F_n(x) - \Phi(x)| \leq \frac{AL_n \ln \frac{1}{L_n}}{1 + x^2}, \quad n \geq 1.$$

Thus

$$\int_{-\infty}^{\infty} |F_n(x) - \Phi(x)|^r dx \leq \left( AL_n \ln \frac{1}{L_n} \right)^r \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^r},$$

which completes the proof, since  $r > \frac{1}{2}$  and  $L_n \rightarrow 0$  as  $n \rightarrow \infty$ . □

PROOF OF THEOREM 2.2. For  $L_n = \mathcal{L}(F_n, \Phi)$ , assume again without loss of generality that  $L_n < e^{-1/2}$  for all  $n \geq 1$ . Let  $0 < p_1 < p$  be such that  $r > \frac{1}{p_1}$ . Then, by Theorem 1.2,

$$|F_n(x) - \Phi(x)| \leq \frac{\lambda_{p_1,n} + c_{p_1} L_n (\ln \frac{1}{L_n})^{p_1/2}}{1 + |x|^{p_1}}, \tag{2.4}$$

where

$$\lambda_{p_1,n} = \left| \int_{-\infty}^{\infty} |x|^{p_1} dF_n(x) - \int_{-\infty}^{\infty} |x|^{p_1} d\Phi(x) \right|.$$

We raise both sides of inequality (2.4) to the power  $r$ , then integrate the result over  $-\infty < x < \infty$ , and observe that the integral on the right hand side is finite, since  $r > \frac{1}{p_1}$ . Now, the coefficient in front of the integral tends to zero, since  $L_n = o(1)$  and  $\lambda_{p_1,n} = o(1)$  as  $n \rightarrow \infty$ , which follows from Corollary 7 of Section 8.1 in CHOW and TEICHER [6].

For the sake of completeness, we state here the necessary part of the latter result. □

**Theorem 2.3.** *Let  $\{F_n\}$  be a sequence of distribution functions such that  $F_n \xrightarrow{w} F$  as  $n \rightarrow \infty$ . If condition (2.3) holds, then*

$$\int_{-\infty}^{\infty} |x|^s dF_n(x) \rightarrow \int_{-\infty}^{\infty} |x|^s dF(x), \quad n \rightarrow \infty,$$

for all  $s < p$ .

*Remark 2.1.* It is an interesting observation that Theorem 2.3, with  $F = \Phi$ , is equivalent in a certain sense to the global version of the central limit theorem.<sup>1</sup> To avoid unnecessary technicalities, let  $p = 2$  and

$$\int_{-\infty}^{\infty} x dF_n = 0, \quad \int_{-\infty}^{\infty} x^2 dF_n = 1$$

for all  $n \geq 1$ . Assume that  $F_n \xrightarrow{w} \Phi$  and denote by  $L_n$  the Lévy distance between  $F_n$  and  $\Phi$ . Then, eventually,  $L_n < e^{-1/2}$ , and one can apply Corollary 1.1. Moreover

$$|F_n^*(x) - \Phi^*(x)| \leq \frac{2AL_n \ln \frac{1}{L_n}}{1 + x^2}, \quad x \geq 0, \tag{2.5}$$

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<sup>1</sup>The authors are grateful to Professor V. V. Buldygin for this remark.

where

$$F^*(x) = F(-x) + 1 - F(x), \quad x > 0,$$

for any distribution function  $F$ . For  $0 < s < 2$ , multiply inequality (2.5) by  $|x|^{s-1}$  and integrate the result over  $(-\infty, \infty)$  to obtain

$$\lambda_{s,n} = O(1) \int_{-\infty}^{\infty} |x|^{s-1} |F_n^*(x) - \Phi^*(x)| dx = o(1) \int_{-\infty}^{\infty} \frac{|x|^{s-1}}{1+x^2} dx, \quad n \rightarrow \infty,$$

where  $\lambda_{s,n}$  is defined by (1.5) with  $p = s$  and  $F = F_n$ . Since  $s < 2$ , the integral on the right-hand side is finite, whence we obtain Theorem 2.3 with  $p = 2$  and  $F = \Phi$ .

*Remark 2.2.* Another interesting question concerning Theorem 2.3 is whether or not the case of  $F = \Phi$  has any special feature compared to the general case. For example, is it true that Theorem 2.3 holds for  $s = p$ , too, if  $F = \Phi$ ? To answer this and several other questions, we consider the following general construction.

Let  $\{a_n\}$  be an increasing sequence such that  $a_n \rightarrow \infty$  as  $n \rightarrow \infty$  and let  $b_n > a_n$ . Define  $h_n (> 0)$  as the solution of the equation

$$\int_{-a_n}^{a_n} \varphi(x) dx + 2h_n(b_n - a_n) = 1, \tag{2.6}$$

where  $\varphi$  is the standard normal probability density. Note that

$$h_n(b_n - a_n) = 1 - \Phi(a_n) \rightarrow 0, \quad n \rightarrow \infty. \tag{2.7}$$

Now let  $\varphi_n$  be the probability density defined by

$$\varphi_n(x) = h_n \mathbb{I}_{(-b_n, -a_n)}(x) + \varphi(x) \mathbb{I}_{[-a_n, a_n]}(x) + h_n \mathbb{I}_{(a_n, b_n)}(x),$$

where  $\mathbb{I}_A$  stands for the indicator function of a set  $A$ . This is a probability density, indeed, in view of (2.6). Moreover,  $\varphi_n$  is symmetric and has finite support, so that all its moments exist and all its odd moments are zero. The second moment of  $\varphi_n$  is given by

$$\sigma_n^2 \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x^2 \varphi_n(x) dx = 2h_n \frac{b_n^3 - a_n^3}{3} + \int_{-a_n}^{a_n} x^2 \varphi(x) dx. \tag{2.8}$$

Denote by  $F_n$  the distribution function generated by  $\varphi_n$ . Note that  $F_n \xrightarrow{w} \Phi$ . Indeed, let  $x$  be fixed and  $n_0$  be such that  $-a_n < x < a_n$ ,  $n \geq n_0$ . Then, by (2.7),

$$F_n(x) = h_n(b_n - a_n) + \int_{-a_n}^x \varphi(u) du \rightarrow \Phi(x), \quad n \rightarrow \infty,$$

which means weak convergence.

Via the construction above, we make the following observations.

*Example 2.1.* (There exists a sequence  $\{F_n\}$  such that  $F_n \xrightarrow{w} \Phi$ , but  $\sigma_n^2 \rightarrow \infty$ .) We define  $\varphi_n$  as above and choose  $a_n = n$ ,  $b_n^2 = n/(1 - \Phi(n))$ . Note that, in view of the well-known inequality  $1 - \Phi(x) < \varphi(x)/x$  ( $x > 0$ ), we have  $b_n^2 > n^2$ . Now,  $F_n \xrightarrow{w} \Phi$ , but, by (2.7) and (2.8),

$$\sigma_n^2 > \frac{2}{3} h_n (b_n - a_n) b_n^2 = \frac{2}{3} n \rightarrow \infty.$$

*Example 2.2.* (Given  $c \geq 1$ , there exists  $\{F_n\}$  such that  $F_n \xrightarrow{w} \Phi$  and  $\sigma_n^2 \rightarrow c \geq 1$ .) In case of  $c > 1$ , we define  $\varphi_n$  as above, with  $a_n = n$ ,  $b_n^2 = \beta_n/(1 - \Phi(n))$ , where  $\beta_n \rightarrow \frac{3}{2}(c - 1)$ . Note that, since  $\frac{3}{2}(c - 1) > 0$ , from Feller's inequality and the exponential decrease of  $\varphi(n)$ , we have  $b_n^2 > n^2$  for (say)  $n \geq n_0$ , so that we can start our sequence  $\{F_n\}$  at  $n_0$ . Then  $F_n \xrightarrow{w} \Phi$ , but, by (2.7) and (2.8),

$$\sigma_n^2 = \frac{2}{3} h_n (b_n - a_n) (b_n^2 + a_n b_n + a_n^2) + \int_{-a_n}^{a_n} x^2 \varphi(x) dx = \frac{2}{3} \beta_n + o(1) + \int_{-a_n}^{a_n} x^2 \varphi(x) dx,$$

where the right-hand side tends to  $c$  by our choice of  $\beta_n$ .

The case of  $c = 1$  is a trivial one, since we can simply choose  $F_n = \Phi$  for all  $n$ .

*Example 2.3.* (Case of  $c < 1$  is not possible.) Note that, in general, Example 2.2 cannot be extended to the case of  $c < 1$ , since, via partial integration,

$$\int_{-\infty}^{\infty} x^2 dF_n(x) = \int_0^{\infty} 2xF_n^*(x) dx,$$

so that, e.g., in case of  $\int_{-\infty}^{\infty} x dF_n(x) \rightarrow 0$ , the weak convergence of  $F_n$  to  $\Phi$  together with an application of Fatou's lemma implies

$$\begin{aligned} \liminf_{n \rightarrow \infty} \sigma_n^2 &= \liminf_{n \rightarrow \infty} \int_{-\infty}^{\infty} x^2 dF_n(x) \geq \int_0^{\infty} \lim_{n \rightarrow \infty} 2xF_n^*(x) dx \\ &= \int_0^{\infty} 2x\Phi^*(x) dx = \int_{-\infty}^{\infty} x^2 d\Phi(x) = 1. \end{aligned}$$

*Example 2.4.* (There exists  $\{F_n\}$  such that  $F_n \xrightarrow{w} \Phi$ , but  $\sigma_n^2$  does not converge.) This can, e.g. be verified by the same construction as in Example 2.2, but with  $\beta_n$  that has no finite or infinite limit.

Let

$$\mu_s = \int_{-\infty}^{\infty} |x|^s \varphi(x) dx, \quad \mu_{s,n} = \int_{-\infty}^{\infty} |x|^s \varphi_n(x) dx.$$

*Example 2.5.* (There exists  $\{F_n\}$  such that  $F_n \xrightarrow{w} \Phi$  and  $\mu_{s,n} \rightarrow \mu_s$  for all  $s < 2$ , but  $\sigma_n^2 \rightarrow \infty$ .) Using the same construction as in Example 2.1, we have

$$\begin{aligned} \mu_{s,n} &= \frac{2}{s+1} h_n(b_n^{s+1} - a_n^{s+1}) + \int_{-a_n}^{a_n} |x|^s \varphi(x) dx \\ &= \frac{2}{s+1} (1 - \Phi(n)) \frac{b_n^{s+1} - a_n^{s+1}}{b_n - a_n} + \int_{-a_n}^{a_n} |x|^s \varphi(x) dx. \end{aligned}$$

Since  $a_n/b_n = \sqrt{n(1 - \Phi(n))} = o(1)$  we have  $b_n - a_n \sim b_n$  and  $b_n^{s+1} - a_n^{s+1} \sim b_n^{s+1}$ , whence

$$(1 - \Phi(n)) \frac{b_n^{s+1} - a_n^{s+1}}{b_n - a_n} \sim (1 - \Phi(n)) b_n^s = n^{s/2} (1 - \Phi(n))^{1-s/2} \rightarrow 0$$

for  $s < 2$ . This proves that  $\mu_{s,n} \rightarrow \mu_s$  for  $s < 2$ , but  $\sigma_n^2 = \mu_{2,n} \rightarrow \infty$ .

*Example 2.6.* (There exists  $\{F_n\}$  such that  $F_n \xrightarrow{w} \Phi$ ,  $\sigma_n^2 \rightarrow \infty$ , even  $\mu_{s,n} \rightarrow \infty$  for all  $s > 0$ .) We use the same construction as in Example 2.2, but with  $\beta_n = e^{1/(1-\Phi(n))}$ . As in Example 2.5, the limit of  $\mu_{s,n}$  is determined by the expression

$$(1 - \Phi(n)) b_n^s = (1 - \Phi(n))^{1-s} e^{s/(1-\Phi(n))},$$

which tends to infinity as  $n \rightarrow \infty$ .

We conclude this section by providing a “global version” of Theorem 2.3 as follows.

**Theorem 2.4.** Assume that  $\{F_n\}$  is a sequence of distribution functions such that  $F_n \xrightarrow{w} \Phi$  as  $n \rightarrow \infty$ . If condition (2.3) holds, then

$$\int_0^\infty x^{s-1} |F_n^*(x) - \Phi^*(x)| dx \rightarrow 0, \quad n \rightarrow \infty,$$

for all  $s < p$ , where  $F^*(x)$  denotes the tail of a distribution function  $F$  as before.

Clearly, via partial integration, the conclusion of Theorem 2.4 is stronger than that of Theorem 2.3.

PROOF OF THEOREM 2.4. First note that  $F_n^*(x) \rightarrow \Phi^*(x)$  for all  $x > 0$ , and this convergence is uniform in any interval  $(0, a)$ . Thus

$$\int_0^a x^{s-1} |F_n^*(x) - \Phi^*(x)| dx \rightarrow 0$$

for all  $a > 0$ , whence we conclude that there exists a sequence  $\{a_n\}$  such that  $a_n \rightarrow \infty$  and

$$\int_0^{a_n} x^{s-1} |F_n^*(x) - \Phi^*(x)| dx \rightarrow 0.$$

Now,

$$\begin{aligned} \int_0^\infty x^{s-1} |F_n^*(x) - \Phi^*(x)| dx &= o(1) + \int_{a_n}^\infty x^{s-1} |F_n^*(x) - \Phi^*(x)| dx \\ &\leq o(1) + a_n^{s-p} \int_{a_n}^\infty x^{p-1} |F_n^*(x) - \Phi^*(x)| dx \leq o(1) + ca_n^{s-p}, \end{aligned}$$

where  $c = \sup_n \int_0^\infty x^{p-1} |F_n^*(x) - \Phi^*(x)| dx$  is finite in view of (2.3). Since  $s < p$ , this completes the proof of Theorem 2.4.  $\square$

### 3. Complete convergence

The concept of *complete convergence* has been introduced by HSU and ROBBINS in [10]. By definition, a sequence of random variables  $\{\xi_n\}$  is said to converge completely to zero (denoted by  $\xi_n \xrightarrow{\text{c.c.}} 0$ ,  $n \rightarrow \infty$ ) if not only  $\xi_n \xrightarrow{\text{a.s.}} 0$ , but also  $\xi'_n \xrightarrow{\text{a.s.}} 0$  as  $n \rightarrow \infty$ , for any sequence of random variables  $\{\xi'_n\}$  such that  $\xi_n \stackrel{d}{=} \xi'_n$  for all  $n \geq 1$ , i.e., for  $\xi_n$  and  $\xi'_n$  having the same distribution function for all  $n \geq 1$ .

*Remark 3.1.* HSU and ROBBINS [10] proved that the complete convergence  $\xi_n \xrightarrow{\text{c.c.}} 0$  as  $n \rightarrow \infty$  is equivalent to

$$\sum_{n=1}^{\infty} \mathbf{P}(|\xi_n| \geq \varepsilon) < \infty \quad \text{for all } \varepsilon > 0.$$

On using the notation  $F^*(x)$ ,  $x > 0$ , for the tail of any distribution function  $F$ , the latter condition is equivalent to

$$\sum_{n=1}^{\infty} F_n^*(\varepsilon) < \infty \quad \text{for all } \varepsilon > 0,$$

where  $F_n$  denotes the distribution function of  $\xi_n$ .

Particularly, the convergence  $n^{-1}S_n \xrightarrow{\text{c.c.}} 0$  as  $n \rightarrow \infty$  has been studied in HSU and ROBBINS [10], when  $\{S_n\}$  are partial sums of independent, identically distributed random variables  $\{X_n\}$ .

**Theorem 3.1** (HSU and ROBBINS [10]). *Let  $\{X_n\}$  be a sequence of independent, identically distributed random variables and let  $\{S_n\}$  denote their partial sums. Then  $n^{-1}S_n \xrightarrow{c.c.} 0$  as  $n \rightarrow \infty$ , if and only if condition (2.1) holds.*

If  $F_n$  denotes the distribution function of  $n^{-1/2}S_n$  (with corresponding tails  $F_n^*$ ), Theorem 3.1 can be stated as follows for independent identically distributed terms:

$$\frac{S_n}{n} \xrightarrow{c.c.} 0, \quad n \rightarrow \infty \iff (2.1) \iff \sum_{n=1}^{\infty} F_n^*(\varepsilon\sqrt{n}) < \infty \quad \forall \varepsilon > 0.$$

The following result provides conditions for the convergence of the latter series in the general case.

**Theorem 3.2.** *Let  $\{F_n\}$  be an arbitrary sequence of distribution functions such that  $L_n \leq e^{-1/2}$  for all  $n \geq 1$ . Assume that condition (2.3) holds with some  $p > 2$ . Then*

$$\sum_{n=1}^{\infty} F_n^*(\varepsilon\sqrt{n}) < \infty \quad \text{for all } \varepsilon > 0.$$

Note that we do not assume here that  $F_n \xrightarrow{w} \Phi$  as  $n \rightarrow \infty$ .

PROOF OF THEOREM 3.2. We make use of Theorem 1.2 again. Let  $L_n$  be the Lévy distance between  $F_n$  and  $\Phi$ . Then, by (1.6), for all  $n \geq 1$  and  $\varepsilon > 0$ ,

$$|F_n^*(\varepsilon\sqrt{n}) - \Phi^*(\varepsilon\sqrt{n})| \leq 2 \cdot \frac{\lambda_{p,n} + c_p L_n (\ln \frac{1}{L_n})^{p/2}}{1 + (\varepsilon\sqrt{n})^p},$$

where  $\lambda_{p,n}$  corresponds to the constant in (1.5) with  $F$  being replaced by  $F_n$ . Since  $\sup_n \lambda_{p,n} < \infty$ ,  $\sup_n L_n (\ln \frac{1}{L_n})^{p/2} < \infty$ , and  $p > 2$ ,

$$\sum_{n=1}^{\infty} |F_n^*(\varepsilon\sqrt{n}) - \Phi^*(\varepsilon\sqrt{n})| < \infty \quad \text{for all } \varepsilon > 0.$$

Moreover,

$$\sum_{n=1}^{\infty} \Phi^*(\varepsilon\sqrt{n}) < \infty \quad \text{for all } \varepsilon > 0,$$

so that an application of the triangle inequality completes the proof. The latter result can be obtained by a straightforward computation:

$$\sum_{n=1}^{\infty} \Phi^*(\varepsilon\sqrt{n}) = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \int_{\varepsilon\sqrt{k}}^{\varepsilon\sqrt{k+1}} \varphi^*(t) dt = \sum_{k=1}^{\infty} k \int_{\varepsilon\sqrt{k}}^{\varepsilon\sqrt{k+1}} \varphi^*(t) dt$$

where  $\varphi^*$  is the density of  $\Phi^*$ . This implies that

$$\sum_{n=1}^{\infty} \Phi^*(\varepsilon\sqrt{n}) \leq \varepsilon^{-2} \int_{\varepsilon}^{\infty} t^2 \varphi^*(t) dt \leq \varepsilon^{-2}. \quad \square$$

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