

## SLLN for random fields under conditions on the bivariate dependence structure

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*Dedicated to the 100<sup>th</sup> anniversary of the birthday of Béla Gyires*

**Abstract.** We find the necessary and sufficient conditions for the strong law of large numbers for families of dependent random variables. We consider fields of random variables with some conditions imposed on the dependence structure described in terms of bivariate copulas.

### 1. Introduction

The strong law of large numbers (SLLN) is one of the most important theorems of probability theory and mathematical statistics. In its classical version – the Kolmogorov’s SLLN for independent and identically distributed (i.i.d.) random variables states that the necessary and sufficient condition for the almost sure convergence of arithmetic means of random variables (r.v.’s) is the existence of the first moment. This theorem was extended and generalized in different directions. ETEMADI (cf. [3]) weakened the assumption of independence and proved that the SLLN holds for pairwise independent random variables. This condition was further relaxed by MATUŁA (cf. [11]) to pairwise negatively quadrant dependent sequences and to certain classes of asymptotically quadrant sub-independent r.v.’s (cf. [12]).

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The other way of generalizations of the classical results, arising from the applications, is the SLLN for random fields. The first results of this kind were obtained by SMYTHE (cf. [16], [17]) and GUT (cf. [5]) in the case of independence. For  $r$ -dimensional random fields of independent r.v.'s the necessary and sufficient condition for the SLLN is the finiteness of the moment  $E|X_1| \log_+^{r-1} |X_1|$ , while in the case of the so-called sectorial convergence is the finiteness of  $E|X_1|$  (cf. [6]). A more general approach to sectorial convergence has been recently studied by INDLEKOFER and KLESOV (cf. [8]).

Many authors investigated the sufficient conditions for the SLLN for dependent random fields (cf. [1], [7]), but there are few results under conditions on the dependence (independence) in pairs (cf. [2], [4], [9]). The aim of this paper is to find the necessary and sufficient conditions for the SLLN for fields of r.v.'s which satisfy some conditions on the dependence structure for pairs of r.v.'s.

The convergence of random fields may have different meanings, therefore we shall begin with introducing some notation. Let  $\mathbb{N}^r$ ,  $r \geq 1$  be the set of positive integer  $r$ -dimensional lattice points with the usual partial order  $\preceq$ , for  $\mathbf{m} = (m_1, \dots, m_r)$  and  $\mathbf{n} = (n_1, \dots, n_r)$  we shall write  $\mathbf{m} \preceq \mathbf{n}$  iff  $m_i \leq n_i$  for  $i = 1, \dots, r$  and  $\mathbf{m} \not\preceq \mathbf{n}$  iff  $m_i > n_i$  for some  $i = 1, \dots, r$ . Further let  $|\mathbf{n}| = \prod_{i=1}^r n_i$  and  $\|\mathbf{n}\| = \max_{1 \leq i \leq r} |n_i|$ . For  $\theta \in (0, 1)$  we define the  $r$ -dimensional sector in the following way

$$\mathcal{S}_\theta^r = \left\{ (i_1, \dots, i_r) \in \mathbb{N}^r : \theta < \frac{i_l}{i_k} < \frac{1}{\theta}, \text{ for all } l, k = 1, \dots, r \right\},$$

we shall also write  $\mathcal{S}_0^r = \mathbb{N}^r$ . In the case  $r = 1$  we have  $\mathcal{S}_\theta^r = \mathbb{N}$ . For  $\mathbf{n} \in \mathcal{S}_\theta^r$  let us also introduce the following notation:  $\mathcal{S}_\theta^r(\mathbf{n}) = \mathcal{S}_\theta^r \cap (\mathbf{n})$ ,  $M_\theta(\mathbf{n}) = \text{Card}(\mathcal{S}_\theta^r(\mathbf{n}))$ ,  $\mathcal{S}_\theta^r(\mathbf{k}, \mathbf{n}) = \mathcal{S}_\theta^r(\mathbf{n}) \setminus (\mathbf{k})$ ,  $\mathbf{k} \preceq \mathbf{n}$  and  $\widehat{\mathcal{S}}_\theta^r(\mathbf{n}) = \mathcal{S}_\theta^r \setminus (\mathbf{n})$ , where  $(\mathbf{n}) = \{\mathbf{k} \in \mathbb{N}^r : \mathbf{k} \preceq \mathbf{n}\}$ . Moreover let  $\mathcal{S}_\theta^r(|\mathbf{n}|) = \{\mathbf{i} \in \mathcal{S}_\theta^r : |\mathbf{i}| \leq |\mathbf{n}|\}$ . We are going to study the convergence of sequences  $\{a_{\mathbf{n}}, \mathbf{n} \in \mathcal{S}_\theta^r\}$  and we shall write

$$a_{\mathbf{n}} \rightarrow a, \quad \text{as } \mathbf{n} \rightarrow \infty \text{ in } \mathcal{S}_\theta^r$$

iff for all  $\varepsilon > 0$  there exists  $\mathbf{n}_0 \in \mathcal{S}_\theta^r$  such that for all  $\mathbf{n} \not\preceq \mathbf{n}_0$  we have  $|a_{\mathbf{n}} - a| < \varepsilon$ . Therefore  $\mathbf{n} \rightarrow \infty$  means  $|\mathbf{n}| \rightarrow \infty$  or equivalently  $\|\mathbf{n}\| \rightarrow \infty$ . For the almost sure convergence of random sequences indexed by the elements of  $\mathcal{S}_\theta^r$  we need to introduce the event

$$\{A_{\mathbf{n}}, \text{ i.o. } \mathbf{n} \in \mathcal{S}_\theta^r\} = \bigcap_{\mathbf{n} \in \mathcal{S}_\theta^r} \bigcup_{\mathbf{k} \in \widehat{\mathcal{S}}_\theta^r(\mathbf{n})} A_{\mathbf{k}},$$

where  $\{A_{\mathbf{n}}, \mathbf{n} \in \mathcal{S}_\theta^r\}$  is a family of events on the same probability space. It is easy to see that if  $\{\xi_{\mathbf{n}}, \mathbf{n} \in \mathcal{S}_\theta^r\}$  is a random field, then  $\xi_{\mathbf{n}} \rightarrow 0$  almost surely as  $\mathbf{n} \rightarrow \infty$  in  $\mathcal{S}_\theta^r$  iff  $P(|\xi_{\mathbf{n}}| \geq \varepsilon, \text{ i.o. } \mathbf{n} \in \mathcal{S}_\theta^r) = 0$ , for every  $\varepsilon > 0$ . Thus the generalizations of the Borel–Cantelli lemmas for dependent events with multidimensional indices will be the main tools in our investigations.

Let  $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^r\}$  be a random field, we are going to find the necessary and sufficient conditions for

$$\frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \in \mathcal{S}_\theta^r(\mathbf{n})} (X_{\mathbf{k}} - m_{\mathbf{k}}) \longrightarrow 0, \text{ almost surely, as } \mathbf{n} \rightarrow \infty \text{ in } \mathcal{S}_\theta^r$$

where  $m_{\mathbf{k}} = EX_{\mathbf{k}}I[|X_{\mathbf{k}}| \leq |\mathbf{k}|]$ , as well as for

$$\frac{1}{M_\theta(\mathbf{n})} \sum_{\mathbf{k} \in \mathcal{S}_\theta^r(\mathbf{n})} X_{\mathbf{k}} \longrightarrow c, \text{ almost surely, as } \mathbf{n} \rightarrow \infty \text{ in } \mathcal{S}_\theta^r,$$

where  $c$  is some constant. We consider random fields of dependent r.v.’s and we shall impose some conditions on the bivariate dependence structure.

In recent years the dependence between r.v.’s has been often described in terms of the so-called copula functions. Copulas are used in stochastic modeling in financial and actuarial mathematics. Let us recall (cf. [14]) that the bivariate copula is a function  $C : [0, 1]^2 \rightarrow [0, 1]$  such that  $C(u, 0) = C(0, v) = 0$ ,  $C(u, 1) = u$ ,  $C(1, v) = v$  and  $C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$  for  $0 \leq u_1 \leq u_2 \leq 1$  and  $0 \leq v_1 \leq v_2 \leq 1$ . For any r.v.’s  $X$  and  $Y$  with distribution functions  $F_X(x)$  and  $F_Y(y)$ , there exists a copula  $C_{X,Y}(u, v)$  such that

$$P(X \leq x, Y \leq y) = C_{X,Y}(F_X(x), F_Y(y)),$$

by the Sklar’s theorem (cf. Theorem 2.3.3 in [14]) this function is uniquely determined for  $(u, v) \in \text{Ran}(F_X) \times \text{Ran}(F_Y)$ .

In this paper we study fields of random variables  $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^r\}$  with the bivariate copulas satisfying the following condition

$$C_{X_{\mathbf{i}}, X_{\mathbf{j}}}(u, v) - uv \leq q_{\mathbf{i}, \mathbf{j}} uv(1 - u)(1 - v) \tag{1}$$

for  $(u, v) \in \text{Ran}(F_{X_{\mathbf{i}}}) \times \text{Ran}(F_{Y_{\mathbf{j}}})$  and  $\mathbf{i} \neq \mathbf{j}$  with  $q_{\mathbf{i}, \mathbf{j}} \geq 0$ . Let us observe that from (1) it follows that

$$\begin{aligned} P(X_{\mathbf{i}} \leq s, X_{\mathbf{j}} \leq t) - P(X_{\mathbf{i}} \leq s)P(X_{\mathbf{j}} \leq t) \\ \leq q_{\mathbf{i}, \mathbf{j}} P(X_{\mathbf{i}} \leq s)P(X_{\mathbf{j}} \leq t)P(X_{\mathbf{i}} > s)P(X_{\mathbf{j}} > t). \end{aligned} \tag{2}$$

Let us observe that for pairwise negatively quadrant dependent (in particular pairwise independent) r.v.'s the condition (1) holds, with  $q_{i,j} = 0$ . The other examples of copulas satisfying (1) comprise Farlie–Gumbel–Morgerstern (FGM), Ali–Mikhail–Haq and the Plackett families of copulas (cf. [12]). The random fields considered in our paper are related to asymptotically quadrant independent (AQI) and quadrant sub-independent (AQSI) r.v.'s or fields, which were studied in [9], however in this paper  $q_{ij}$  depends on  $\|\mathbf{i} - \mathbf{j}\|$ . Furthermore the results in [9] were established for bounded or square-integrable random fields and did not provide necessary conditions. Therefore our results cannot be obtained from the previously known ones.

## 2. Main results and proofs

**Theorem 2.1.** *Let  $\{X_{\mathbf{n}}, \mathbf{n} \in \mathcal{S}_{\theta}^r\}$  be a family of equidistributed random variables satisfying condition (1) and such that*

$$\sum_{\mathbf{j} \in \mathcal{S}_{\theta}^r} \sum_{\mathbf{i} \in \mathcal{S}_{\theta}^r(\|\mathbf{j}\|), \mathbf{i} \neq \mathbf{j}} |\mathbf{j}|^{-2} q_{i,j} < \infty \text{ and } \sup_{\mathbf{k}, \mathbf{j} \in \mathcal{S}_{\theta}^r} q_{\mathbf{k}, \mathbf{j}} < \infty. \quad (3)$$

Then, the following conditions are equivalent:

$$\frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \in \mathcal{S}_{\theta}^r(\mathbf{n})} (X_{\mathbf{k}} - m_{\mathbf{k}}) \rightarrow 0, \text{ almost surely as } \mathbf{n} \rightarrow \infty \text{ in } \mathcal{S}_{\theta}^r, \quad (4)$$

where  $m_{\mathbf{k}} = EX_{\mathbf{k}}I[|X_{\mathbf{k}}| \leq |\mathbf{k}|]$ ,

$$\begin{aligned} E|X_{\mathbf{1}}|(\log_+ |X_{\mathbf{1}}|^{r-1}) < \infty, & \text{ if } \theta = 0 \\ E|X_{\mathbf{1}}| < \infty, & \text{ if } \theta \in (0, 1). \end{aligned} \quad (5)$$

PROOF. For  $a < b$  let us define  $\varphi_{a,b}(t) = aI[t \leq a] + tI[a < t < b] + bI[t \geq b]$ . Let us begin with the sufficient condition (5)  $\implies$  (4). Obviously

$$\sum_{\mathbf{j} \in \mathcal{S}_{\theta}^r} P(\varphi_{-|\mathbf{j}|, |\mathbf{j}|}(X_{\mathbf{j}}) \neq X_{\mathbf{j}}) = \sum_{\mathbf{j} \in \mathcal{S}_{\theta}^r} P(|X_{\mathbf{j}}| > |\mathbf{j}|) \quad (6)$$

and the r.h.s. of (6) is finite by (5) according to Lemma 2.1 in [5] and Lemma 2.1 in [6]. Therefore it is enough to prove (4) for truncated random field  $\{\varphi_{-|\mathbf{j}|, |\mathbf{j}|}(X_{\mathbf{j}}), \mathbf{j} \in \mathcal{S}_{\theta}^r\}$ . To be exact, we shall prove it for  $X_{\mathbf{j}}' = \varphi_{-|\mathbf{j}|, |\mathbf{j}|}^+(X_{\mathbf{j}}) = \varphi_{0, |\mathbf{j}|}(X_{\mathbf{j}})$  and  $X_{\mathbf{j}}'' = \varphi_{-|\mathbf{j}|, |\mathbf{j}|}^-(X_{\mathbf{j}}) = -\varphi_{-|\mathbf{j}|, 0}(X_{\mathbf{j}})$ . Now, let us observe that the family  $\{X_{\mathbf{j}}', \mathbf{j} \in \mathcal{S}_{\theta}^r\}$

satisfies the assumptions (i)–(iii) of Lemma 3.1. To check (iv), let us at first note, that by our assumption (1) and Lemma 3.2 we get for  $\mathbf{j} \neq \mathbf{k}$

$$\begin{aligned} \text{Cov}(X'_\mathbf{j}, X'_\mathbf{k}) &= \int_0^{|\mathbf{j}|} \int_0^{|\mathbf{k}|} [P(X_\mathbf{j} \leq u, X_\mathbf{k} \leq v) - P(X_\mathbf{j} \leq u)P(X_\mathbf{k} \leq v)]dudv \\ &\leq q_{\mathbf{j},\mathbf{k}} \int_0^{|\mathbf{j}|} P(X_\mathbf{j} > u)du \int_0^{|\mathbf{k}|} P(X_\mathbf{k} > v)dv = q_{\mathbf{j},\mathbf{k}} EX'_\mathbf{j} EX'_\mathbf{k} \\ &\leq q_{\mathbf{j},\mathbf{k}} (E|X_1|)^2. \end{aligned}$$

Thus we get

$$\begin{aligned} &\sum_{\mathbf{k} \in \mathcal{S}_\theta^r} \sum_{\mathbf{j} \in \mathcal{S}_\theta^r(|\mathbf{k}|)} |\mathbf{k}|^{-2} \text{Cov}^+(X'_\mathbf{j} I[X'_\mathbf{j} \leq |\mathbf{j}|], X'_\mathbf{k} I[X'_\mathbf{k} \leq |\mathbf{k}|]) \\ &\leq (E|X_1|)^2 \sum_{\mathbf{j} \in \mathcal{S}_\theta^r} \sum_{\mathbf{i} \in \mathcal{S}_\theta^r(|\mathbf{j}|), \mathbf{i} \neq \mathbf{j}} |\mathbf{j}|^{-2} q_{\mathbf{i},\mathbf{j}} + \sum_{\mathbf{j} \in \mathcal{S}_\theta^r} |\mathbf{j}|^{-2} \text{Var}(X'_\mathbf{j}) < \infty \end{aligned}$$

by (3) and since  $\sum_{\mathbf{j} \in \mathcal{S}_\theta^r} |\mathbf{j}|^{-2} \text{Var}(X'_\mathbf{j})$  is bounded by  $E|X_1|(\log_+ |X_1|^{r-1})$ , if  $\theta = 0$  and by  $E|X_1|$  in the sectorial case  $\theta \in (0, 1)$  (cf. Lemma 2.2 and its proof in [5]). Thus, by Lemma 3.1 we have

$$\frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \in \mathcal{S}_\theta^r(\mathbf{n})} (X'_\mathbf{k} - EX'_\mathbf{k}) \rightarrow 0, \text{ almost surely as } \mathbf{n} \rightarrow \infty \text{ in } \mathcal{S}_\theta^r$$

similar result holds for  $\{X''_\mathbf{j}, \mathbf{j} \in \mathcal{S}_\theta^r\}$ , by the moment assumption (5) we have  $|\mathbf{n}|P(|X_1| > |\mathbf{n}|) \rightarrow 0$  as  $\mathbf{n} \rightarrow \infty$  and the proof of sufficiency is completed.

To prove necessity (4)  $\Rightarrow$  (5), let us observe that by the standard arguments we get  $X_\mathbf{n}/|\mathbf{n}| \rightarrow 0$  almost surely as  $\mathbf{n} \rightarrow \infty$  in  $\mathcal{S}_\theta^r$ . Thus, by Lemma 3.3,  $\sum_{\mathbf{k} \in \mathcal{S}_\theta^r} P(|X_\mathbf{k}| \geq |\mathbf{k}|) < \infty$ , what gives (5) (cf. [5], [6]).  $\square$

Let us go to the more classical version of the SLLN and prove the following strong law for pairwise dependent random variables satisfying condition (1).

**Theorem 2.2.** *Let  $\{X_\mathbf{n}, \mathbf{n} \in \mathcal{S}_\theta^r\}$  be a family of equidistributed random variables satisfying condition (1) and (3). Then the condition (5) is equivalent to*

$$\frac{1}{M_\theta(\mathbf{n})} \sum_{\mathbf{k} \in \mathcal{S}_\theta^r(\mathbf{n})} X_\mathbf{k} \rightarrow c, \text{ almost surely as } \mathbf{n} \rightarrow \infty \text{ in } \mathcal{S}_\theta^r, \text{ for some constant } c. \quad (7)$$

If (5) holds then  $c = EX_1$ .

PROOF. From (7) it follows that  $\frac{1}{M_\theta(\mathbf{n})} \sum_{\mathbf{k} \in \mathcal{S}_\theta^r(\mathbf{n})} (X_\mathbf{k} - c) \rightarrow 0$  almost surely, thus  $\frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \in \mathcal{S}_\theta^r(\mathbf{n})} (X_\mathbf{k} - c) \rightarrow 0$  and consequently  $X_\mathbf{n}/|\mathbf{n}| \rightarrow 0$  almost surely as  $\mathbf{n} \rightarrow \infty$  in  $\mathcal{S}_\theta^r$ . Since that (5) follows from Lemma 3.3 as in the proof of

Theorem 2.1. Conversely from (5), by Theorem 2.1, we get  $\frac{1}{M_\theta(\mathbf{n})} \sum_{\mathbf{k} \in \mathcal{S}_\theta^r(\mathbf{n})} (X_{\mathbf{k}} - m_{\mathbf{k}}) \rightarrow 0$ . Furthermore  $EX_{\mathbf{1}}I[|X_{\mathbf{1}}| > |\mathbf{n}|] \rightarrow 0$ , thus the generalization of the Toeplitz lemma (cf. Lemma 2 in [13]), with  $w(\mathbf{m}, \mathbf{k}) = \frac{1}{M_\theta(\mathbf{m})}$  for  $\mathbf{k} \in \mathcal{S}_\theta^r(\mathbf{m})$  and  $w(\mathbf{m}, \mathbf{k}) = 0$  otherwise, implies (7).  $\square$

*Remark 2.1.* Let us observe that in general, the convergence of the series

$$\sum_{\mathbf{j} \in \mathcal{S}_\theta^r} \sum_{\mathbf{i} \in \mathcal{S}_\theta^r(|\mathbf{j}|), \mathbf{i} \neq \mathbf{j}} |\mathbf{j}|^{-2} q_{\mathbf{i}, \mathbf{j}} < \infty$$

does not imply  $\sup_{\mathbf{k}, \mathbf{j} \in \mathcal{S}_\theta^r} q_{\mathbf{k}, \mathbf{j}} < \infty$ . Even in the case  $r = 1$  the family  $q_{\mathbf{k}, \mathbf{j}}$  may contain divergent subsequences.

*Remark 2.2.* Let us point out some special forms of  $q_{\mathbf{i}, \mathbf{j}}$  for which (3) holds:

- (1)  $\sup_{\mathbf{i} \in \mathcal{S}_\theta^r(|\mathbf{j}|)} q_{\mathbf{i}, \mathbf{j}} \leq \frac{C}{\log_+^{1+\delta} |\mathbf{j}|}$  for  $\theta \in (0, 1)$ ,
- (2)  $\sup_{\mathbf{i} \in \mathcal{S}_\theta^r(|\mathbf{j}|)} q_{\mathbf{i}, \mathbf{j}} \leq \frac{C}{\log_+^{r+\delta} |\mathbf{j}|}$  for  $\theta = 0$ ,

for some constants  $C, \delta > 0$  and any  $\mathbf{j} \in \mathcal{S}_\theta^r$ . Applying well known methods of summation of multiple series in both cases we obtain

$$\sum_{\mathbf{j} \in \mathcal{S}_\theta^r} \sum_{\mathbf{i} \in \mathcal{S}_\theta^r(|\mathbf{j}|), \mathbf{i} \neq \mathbf{j}} |\mathbf{j}|^{-2} q_{\mathbf{i}, \mathbf{j}} \leq C \sum_{k=2}^{\infty} \frac{1}{k \log_+^{1+\delta} k} < \infty.$$

*Remark 2.3.* If  $q_{\mathbf{i}, \mathbf{j}} = q(\|\mathbf{i} - \mathbf{j}\|)$ , then by using

$$d_\theta(k) = \text{Card} \{ \mathbf{n} \in \mathcal{S}_\theta^r : |\mathbf{n}| = k \} = o(k^\delta),$$

we get that  $\sum_{k=1}^{\infty} q(kt)/k^{1-2\delta} < \infty$ , for some  $\delta > 0$  is a sufficient condition for (3).

Often it is assumed that  $q_{\mathbf{i}, \mathbf{j}}$  depends only on the distance of  $\mathbf{i}$  and  $\mathbf{j}$  i.e.  $q_{\mathbf{i}, \mathbf{j}} = q(\|\mathbf{i} - \mathbf{j}\|)$ . In this case the condition (3) may be written as

$$\sum_{\mathbf{j} \in \mathcal{S}_\theta^r} \sum_{\mathbf{i} \in \mathcal{S}_\theta^r: |\mathbf{i}| \leq |\mathbf{j}|} |\mathbf{j}|^{-2} q(\|\mathbf{i} - \mathbf{j}\|) < \infty. \tag{8}$$

and in the one-dimensional case we get the following generalization of the main result in [12].

**Corollary 2.1.** *Let  $\{X_n, n \in \mathbb{N}\}$  be a sequence of equidistributed random variables satisfying condition (1) and such that  $\sum_{n=1}^{\infty} q(n)/n < \infty$ . Then the following conditions are equivalent:*

$$\frac{1}{n} \sum_{k=1}^n X_k \rightarrow c, \text{ almost surely, as } n \rightarrow \infty \text{ for some constant } c \tag{9}$$

$$E|X_{\mathbf{1}}| < \infty. \tag{10}$$

If  $E|X_{\mathbf{1}}| < \infty$ , then  $c = EX_{\mathbf{1}}$ .

*Example 1.* In a similar way as in [12] we can construct a random field  $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^r\}$  of r.v.'s satisfying (1) with the same distribution function  $F(x)$  by introducing a consistent family of finite-dimensional FGM distributions. The joint distribution of  $X_{\mathbf{i}_1}, \dots, X_{\mathbf{i}_n}$  is given by

$$F_{\mathbf{i}_1, \dots, \mathbf{i}_n}(x_1, \dots, x_n) = \prod_{k=1}^n F(x_k) \left( 1 + \sum_{1 \leq j < k \leq n} a_{\mathbf{i}_j, \mathbf{i}_k} (1 - F(x_j))(1 - F(x_k)) \right)$$

with  $a_{\mathbf{i}_j, \mathbf{i}_k} = \pm A^{-|\mathbf{i}_j| - |\mathbf{i}_k|}$ , for some  $A > 1$  chosen in such a way that

$$\left| \sum_{1 \leq j < k \leq n} a_{\mathbf{i}_j, \mathbf{i}_k} \right| \leq \left| \sum_{\mathbf{i}, \mathbf{j} \in \mathbb{N}^r} \frac{1}{A^{|\mathbf{i}| + |\mathbf{j}|}} \right| \leq \left( \sum_{k \in \mathbb{N}} \frac{d_0(k)}{A^k} \right)^2 \leq 1$$

for any choice of  $\mathbf{i}_1, \dots, \mathbf{i}_n$ , here  $d_0(k) = \text{Card}\{\mathbf{n} \in \mathbb{N}^r : |\mathbf{n}| = k\} = o(k^\delta)$ , for some  $\delta > 0$ . The bivariate distribution of  $X_{\mathbf{i}}, X_{\mathbf{j}}$  is the FGM distribution with the copula of the following form

$$C_{X_{\mathbf{i}}, X_{\mathbf{j}}}(u, v) = uv(1 + a_{\mathbf{ij}}(1 - u)(1 - v))$$

so that we may take  $q_{\mathbf{ij}} = A^{-|\mathbf{i}| - |\mathbf{j}|}$  if  $a_{\mathbf{ij}} > 0$  and 0 otherwise. It is easy to see that in this case, the conditions mentioned in Remark 2.2 are satisfied.

### 3. Auxiliary lemmas

In the first lemma and proofs of our results we shall use the following notation

$$\text{Cov}^+(X, Y) = \max(\text{Cov}(X, Y), 0).$$

**Lemma 3.1.** *Let  $\{X_{\mathbf{n}}, \mathbf{n} \in \mathcal{S}_\theta^r\}$  be a field of nonnegative random variables such that:*

- (i)  $\sup_{\mathbf{k} \in \mathcal{S}_\theta^r} EX_{\mathbf{k}} < \infty$ ,
- (ii)  $\sum_{\mathbf{k} \in \mathcal{S}_\theta^r} P(X_{\mathbf{k}} > |\mathbf{k}|) < \infty$ ,
- (iii)  $\frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \in \mathcal{S}_\theta^r(\mathbf{n})} EX_{\mathbf{k}} I[X_{\mathbf{k}} > |\mathbf{k}|] \rightarrow 0$ , as  $\mathbf{n} \rightarrow \infty$  in  $\mathcal{S}_\theta^r$ ,
- (iv)  $\sum_{\mathbf{k} \in \mathcal{S}_\theta^r} \sum_{\mathbf{j} \in \mathcal{S}_\theta^r(|\mathbf{k}|)} |\mathbf{k}|^{-2} \text{Cov}^+(X_{\mathbf{j}} I[X_{\mathbf{j}} \leq |\mathbf{j}|], X_{\mathbf{k}} I[X_{\mathbf{k}} \leq |\mathbf{k}|]) < \infty$ .

Then

$$\frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \in \mathcal{S}_\theta^r(\mathbf{n})} (X_{\mathbf{k}} - EX_{\mathbf{k}}) \rightarrow 0, \text{ almost surely as } \mathbf{n} \rightarrow \infty \text{ in } \mathcal{S}_\theta^r.$$

PROOF. For  $\theta = 0$  the lemma was obtained by KO et al. [9] in the case  $\min_{1 \leq i \leq r} n_i \rightarrow \infty$ , but their arguments also work when  $|\mathbf{n}| \rightarrow \infty$ . The sectorial case  $\theta \in (0, 1)$  follows from the already proved part by considering the random field  $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^r\}$ , where  $X_{\mathbf{n}} = 0$  if  $\mathbf{n} \in \mathbb{N}^r \setminus \mathcal{S}_\theta^r$ .  $\square$

The next lemma is a version of the Hoeffding lemma (cf. [10]) for continuously truncated random variables. Let us recall that  $\varphi_{a,b}(t) = aI[t \leq a] + tI[a < t < b] + bI[t \geq b]$ .

**Lemma 3.2.** *Let  $(X, Y)$  be any 2-dimensional random vector then*

$$\text{Cov}(\varphi_{a,b}(X), \varphi_{a,b}(Y)) = \int_a^b \int_a^b [P(X \leq u, Y \leq v) - P(X \leq u)P(Y \leq v)] dudv,$$

furthermore

$$E\varphi_{0,a}(X) = \int_0^a P(X \geq u) du.$$

PROOF. Applying Hoeffding equality

$$\text{Cov}(\xi, \eta) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [P(\xi \leq u, \eta \leq v) - P(\xi \leq u)P(\eta \leq v)] dudv$$

to the r.v.'s  $\xi = \varphi_{a,b}(X)$  and  $\eta = \varphi_{a,b}(Y)$  we get the desired conclusion.  $\square$

In the following lemma we shall present an extension of the classical Erdős–Rényi version of the second Borel–Cantelli lemma to families of dependent events indexed by multidimensional indices.

**Lemma 3.3.** *Let  $\{A_{\mathbf{n}} \in \mathcal{F}, \mathbf{n} \in \mathcal{S}_\theta^r\}$  be a family of events on some probability space  $(\Omega, \mathcal{F}, P)$  and  $\{q_{\mathbf{i}, \mathbf{j}}, \mathbf{i}, \mathbf{j} \in \mathcal{S}_\theta^r\}$  a family of positive numbers satisfying the following conditions*

- (i)  $\sum_{\mathbf{k} \in \mathcal{S}_\theta^r} P(A_{\mathbf{k}}) = \infty$ ,
- (ii) *there exists  $\mathbf{n}_0 \in \mathcal{S}_\theta^r$  such that for all  $\mathbf{k}, \mathbf{j} \in \widehat{\mathcal{S}}_\theta^r(\mathbf{n}_0)$*

$$P(A_{\mathbf{k}} \cap A_{\mathbf{j}}) - P(A_{\mathbf{k}})P(A_{\mathbf{j}}) \leq q_{\mathbf{k}, \mathbf{j}}P(A_{\mathbf{k}})P(A_{\mathbf{j}}),$$

- (iii)  $\sup_{\mathbf{k}, \mathbf{j} \in \widehat{\mathcal{S}}_\theta^r(\mathbf{n}_0)} q_{\mathbf{k}, \mathbf{j}} < \infty$ .

Then for any  $\mathbf{n} \not\leq \mathbf{n}_0$

$$P(A_{\mathbf{n}}, \text{ i.o. } \mathbf{n} \in \mathcal{S}_\theta^r) \geq \frac{1}{1 + \sup_{\mathbf{k}, \mathbf{j} \in \widehat{\mathcal{S}}_\theta^r(\mathbf{n})} q_{\mathbf{k}, \mathbf{j}}}.$$

PROOF. By the Chung–Erdős inequality (cf. [15] p. 284), applied to the finite family of events  $\{A_{\mathbf{k}}, \mathbf{k} \in \mathcal{S}_\theta^r(\mathbf{n}, \mathbf{l})\}$ , we have

$$P\left(\bigcup_{\mathbf{k} \in \mathcal{S}_\theta^r(\mathbf{n}, \mathbf{l})} A_{\mathbf{k}}\right) \geq \left(\sum_{\mathbf{k} \in \mathcal{S}_\theta^r(\mathbf{n}, \mathbf{l})} P(A_{\mathbf{k}})\right)^2 / \sum_{\mathbf{k}, \mathbf{j} \in \mathcal{S}_\theta^r(\mathbf{n}, \mathbf{l})} P(A_{\mathbf{k}} \cap A_{\mathbf{j}}).$$

From our assumption (ii), we get

$$\begin{aligned} \sum_{\mathbf{k}, \mathbf{j} \in \mathcal{S}_\theta^r(\mathbf{n}, \mathbf{l})} P(A_{\mathbf{k}} \cap A_{\mathbf{j}}) &\leq \sum_{\mathbf{k}, \mathbf{j} \in \mathcal{S}_\theta^r(\mathbf{n}, \mathbf{l})} (1 + q_{\mathbf{k}, \mathbf{j}}) P(A_{\mathbf{k}})P(A_{\mathbf{j}}) \\ &\leq \left(1 + \sup_{\mathbf{k}, \mathbf{j} \in \mathcal{S}_\theta^r(\mathbf{n}, \mathbf{l})} q_{\mathbf{k}, \mathbf{j}}\right) \left(\left(\sum_{\mathbf{k} \in \mathcal{S}_\theta^r(\mathbf{n}, \mathbf{l})} P(A_{\mathbf{k}})\right)^2 + \sum_{\mathbf{k} \in \mathcal{S}_\theta^r(\mathbf{n}, \mathbf{l})} P(A_{\mathbf{k}})\right). \end{aligned}$$

Now, using the idea of PETROV [15] we derive

$$\begin{aligned} P\left(\bigcup_{\mathbf{k} \in \mathcal{S}_\theta^r(\mathbf{n}, \mathbf{l})} A_{\mathbf{k}}\right) &\geq \left(1 + \sup_{\mathbf{i}, \mathbf{j} \in \mathcal{S}_\theta^r(\mathbf{n}, \mathbf{l})} q_{\mathbf{i}, \mathbf{j}}\right)^{-1} \frac{\left(\sum_{\mathbf{k} \in \mathcal{S}_\theta^r(\mathbf{n}, \mathbf{l})} P(A_{\mathbf{k}})\right)^2}{\left(\sum_{\mathbf{k} \in \mathcal{S}_\theta^r(\mathbf{n}, \mathbf{l})} P(A_{\mathbf{k}})\right)^2 + \sum_{\mathbf{k} \in \mathcal{S}_\theta^r(\mathbf{n}, \mathbf{l})} P(A_{\mathbf{k}})} \end{aligned}$$

and taking the limit over  $\mathbf{l}$  of the both sides, we obtain

$$P\left(\bigcup_{\mathbf{k} \in \widehat{\mathcal{S}}_\theta^r(\mathbf{n})} A_{\mathbf{k}}\right) \geq \frac{1}{1 + \sup_{\mathbf{i}, \mathbf{j} \in \widehat{\mathcal{S}}_\theta^r(\mathbf{n})} q_{\mathbf{i}, \mathbf{j}}}.$$

Now, one can easily see that

$$\bigcap_{\mathbf{n} \in \mathcal{S}_\theta^r} \bigcup_{\mathbf{k} \in \widehat{\mathcal{S}}_\theta^r(\mathbf{n})} A_{\mathbf{k}} = \bigcap_{N=1}^{\infty} \bigcup_{\mathbf{k} \in \widehat{\mathcal{S}}_\theta^r(\mathbf{N})} A_{\mathbf{k}}$$

where  $\mathbf{N} = (N, \dots, N)$ . The sequence of events  $B_N = \bigcup_{\mathbf{k} \in \widehat{\mathcal{S}}_\theta^r(\mathbf{N})} A_{\mathbf{k}}$  is decreasing, hence

$$P(A_{\mathbf{n}}, \text{ i.o. } \mathbf{n} \in \mathcal{S}_\theta^r) = \lim_{N \rightarrow \infty} P\left(\bigcup_{\mathbf{k} \in \widehat{\mathcal{S}}_\theta^r(\mathbf{N})} A_{\mathbf{k}}\right) \geq \frac{1}{1 + \sup_{\mathbf{i}, \mathbf{j} \in \widehat{\mathcal{S}}_\theta^r(\mathbf{n}_0)} q_{\mathbf{i}, \mathbf{j}}}. \quad \square$$

*Remark 3.1.* Under conditions of Lemma 3.3 with (iii) replaced by

$$(iii') \sup_{\mathbf{k}, \mathbf{j} \in \widehat{\mathcal{S}}_{\theta}^r(\mathbf{n})} q_{\mathbf{k}, \mathbf{j}} \rightarrow 0, \text{ as } \mathbf{n} \rightarrow \infty,$$

we have  $P(A_{\mathbf{n}}, \text{i.o. } \mathbf{n} \in \mathcal{S}_{\theta}^r) = 1$

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