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Lorentzian Para-contact submanifolds

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Matsumoto and Mihai [1] introduced the idea of Lorentzian Para-contact structure and studied its several properties. The purpose of the present paper is to initiate the study of Lorentzian Para-contact submanifolds.

1. Introduction

Let us consider an n-dimensional real differentiable manifold of differentiability class C^∞ endowed with a C^∞ vector valued linear function φ, a C^∞ vector field ξ and a C^∞ 1-form η and a Lorentzian metric g satisfying

$$\varphi^2(V) = V + \eta(V)\xi \tag{1.1}$$

$$\eta(\xi) = -1 \tag{1.2}$$

$$g(\varphi U, \varphi V) = g(U,V) + \eta(U)\eta(V) \tag{1.3}$$

$$g(V,\xi) = \eta(V) \tag{1.4}$$

for arbitrary vector fields U and V, then V_n is called a Lorentzian Para-contact manifold and the structure (φ,ξ,η,g) is called a Lorentzian Para-contact structure.

In a Lorentzian para-contact structure the following hold:

$$\varphi\xi = 0, \quad \eta(\varphi V) = 0 \tag{1.5}$$

$$\operatorname{rank}(\varphi) = n-1. \tag{1.6}$$

A Lorentzian para-contact manifold is called Lorentzian Para-Sasakian manifold if

$$(\nabla_U \varphi)(V) = g(U,V)\xi + U\eta(V) + 2\eta(U)\eta(V) \tag{1.7}$$

and

$$\nabla_U \xi = \varphi U \tag{1.8}$$

where ∇_U denotes the covariant differentiation with respect to g.
Let us put

$$\Phi(U,V) = g(\varphi U, V) \tag{1.9}$$

Then the tensor field Φ is symmetric:

$$\Phi(U,V) = \Phi(V,U) \tag{1.10}$$

and

$$\Phi(U,V) = (\nabla_U \eta)(V). \tag{1.11}$$

Definition 1.1. An Lorentzian para-contact manifold will be called an LP–cosymplectic manifold if

$$\nabla_U \varphi = 0 \tag{1.12}$$

Definition 1.2. An Lorentzian Para (LP)-contact manifold will be called an LP–nearly cosymplectic manifold if

$$(a) \quad (\nabla_U \varphi)(U) = 0 \iff (\nabla_U \varphi)(V) + (\nabla_V \varphi)(U) = 0. \tag{1.13}$$

It can be easily seen that on an LP–Cosymplectic manifold $\nabla_U \xi = 0$.

Theorem 1.1. *The Lorentzian para-contact structure on V_n is not unique.*

Proof. Let (φ, ξ, η, g) be a Lorentzian Para-contact structure on V_n. Let ξ' be a nonzero vector field nowhere in the ξ-direction, then we have a non-singular tensor field μ of type $(1,1)$ such that $\mu\xi' = \xi$. If we define a tensor field φ' and a 1–form η' by $\mu\varphi' U = \varphi\mu U$, $\eta'(U) = \eta(\mu U)$, then we have

$$\mu\varphi'^2 U = \varphi\mu\varphi' U = \varphi^2 \mu U = \mu U + \eta(\mu U) = \mu(U + \eta'(U)\xi)$$

yielding

$$\varphi'^2 U = U + \eta'(U)\xi'.$$

Let us define a metric tensor g' by $g'(U,V) = g(\mu U, \mu V)$.
Then $g'(\varphi' U, \varphi' V) = g(\mu\varphi' U, \mu\varphi' V) = g'(U,V) + \eta'(U)\eta'(V)$.
Also $g'(\xi', U) = \eta'(U)$.

Thus $(\varphi', \xi', \eta', g')$ is another Lorentzian Para-contact structure.

Theorem 1.2. *On a nearly LP–Cosymplectic manifold $\nabla_U \xi = 0$.*

Proof. Equation (1.13) (b) is equivalent to

$$(\nabla_U \Phi)(V,W) = (\nabla_V \Phi)(U,W) = 0. \tag{1.14}$$

Equations (1.9) and (1.14) give

$$(\nabla_U \eta)(\varphi V) + (\nabla_V \eta)(\varphi U) = 0. \tag{1.15}$$

Putting ξ for U in (1.14) and using in (1.15) we get $\nabla_U \xi = 0$.

2. Lorentzian Para-contact submanifold

Let V_{2m-1} be a submanifold of V_{2m+1} with the inclusion map $b : V_{2m-1} \to V_{2m+1}$ such that $p \in V_{2m-1}$ goes to $bp \in V_{2m+1}$. The map b induces a linear transformation (Jacobian map) $b_\star : T_{(2m-1)} \to T_{(2m+1)}$ where T_{2m-1} is the tangent space to V_{2m-1} at a point p and T_{2m+1} is the tangent space to V_{2m+1} at a point bp such that

$$(X \text{ in } V_{2m-1} \text{ at } p) \to (b_\star X \text{ in } V_{2m+1} \text{ at } bp)$$

Agreement 2.1. In what follows the equations containing X, Y, Z hold for arbitrary vector fields X, Y, Z in V_{2m-1}.

Let M, N be mutually orthogonal unit vectors normal to V_{2m-1}. If $\tilde{g}$ is an induced metric tensor in V_{2m-1}, we have

$$\begin{array}{lllll} & (a) & g(b_\star X, b_\star Y)\text{ob} = \tilde{g}(X,Y) & (b) & g(b_\star X, M)\text{ob} = 0 \\ (2.1) & (c) & g(b_\star X, N)\text{ob} = 0 & (d) & g(M,N)\text{ob} = 0 \\ & (e) & g(M,M)\text{ob} = g(N,N)\text{ob} = 1. & & \end{array}$$

If D is the induced connection on V_{2m-1}, then we have the Gauss equation

$$V_{b_\star X} b_\star Y = b_\star D_X Y + MH(X,Y) + NK(X,Y) \tag{2.2}$$

where H and K are symmetric bilinear functions in V_{2m-1}. The Weingarten equations in V_{2m-1} are given by

$$\begin{aligned} V_{b_\star X} M &= -b_\star{}'H(X) + l(X)M, \qquad & g('H(X), Y) &\overset{\text{def}}{=} H(X,Y), \\ V_{b_\star X} N &= -b_\star{}'K(X) - l'(X)N, \qquad & g('K(X), Y) &\overset{\text{def}}{=} K(X,Y). \end{aligned}$$

If the second fundamental forms H and K of V_{2m-1} are of the form $H(X,Y) = \mu_1 \tilde{g}(X,Y)$, $K(X,Y) = \mu_2$, $g(X,Y)$ where $\mu_1, \mu_2 = (\operatorname{Tr} b_\star)/n'$ then V_{2m-1} is called totally umbilical. In our case, we take $\mu_1 = \mu_2 = \mu$. If the second fundamental form vanishes identically then V_{2m-1} is said to be totally geodetic. (Yano and Kon [3]).

A submanifold V_{2m-1} of a Lorentzian Para-contact manifold V_{2m+1} is said to be invariant if the structure vector field ξ of V_{2m-1} is tangent to V_{2m+1} and $\varphi(T_X(V_{2m-1}) \subseteq T_X(V_{2m-1})$ where $T_X(V_{2m-1})$ denotes the tangent space of V_{2m-1} at X. On the other hand if $\varphi(T_X(V_{2m-1}) \subseteq T_X(V_{2m-1})^\perp$ for all $X \in V_{2m-1}$, where $T_X(V_{2m-1})^\perp$ is the normal space of V_{2m-1} at X then V_{2m-1} is said to be antiinvariant in V_{2m-1}.

Let us put

$$\begin{aligned} &(a) \quad \varphi(b_\star X) = b_\star X + \alpha(X)M + \nu(X)N \\ &(b) \quad \xi = b_\star \xi + \rho M + \sigma N \\ &(c) \quad \varphi(M) = -b_\star p + \delta N \\ &(d) \quad \varphi(N) = -b_\star q + \theta M \end{aligned} \tag{2.3}$$

Pre-multiplying (2.3) (a) by φ and using (1.1), (2.3) (b), (c), (d) we obtain

$$\begin{aligned} &b_\star X + \eta(b_\star X)(b_\star \xi + \rho M + \sigma N) = b_\star \phi^2 X - b_\star p\alpha(X) \\ &- b_\star q\nu(X) + M(\alpha(\phi(X)) + \theta\nu(X) + N(\nu(\phi(X)) + \delta\alpha(X)) \end{aligned} \tag{2.4}$$

Substituting from (2.3) (a) in

$$g(\varphi(b_\star X), \varphi(b_\star Y)) = g(b_\star X, b_\star Y) + \eta(b_\star X)\eta(b_\star Y)$$

and using (2.1) we obtain

$$\begin{aligned} g(\phi X, \phi Y) = g(X, Y) + (\eta(b_\star X)ob)(\eta(b_\star Y)ob) \\ -\alpha(X)\alpha(Y) - \nu(X)\nu(Y) \end{aligned} \tag{2.5}$$

Equations (2.4) and (2.5) give

$$\varphi^2 X = X + a(X)T$$
$$g(\varphi X, \varphi Y) = g(X, Y) + a(X)a(Y)$$

iff

$$(b_\star X)ob = a(X), \qquad p(\alpha(X)) + q(\nu(X)) = 0$$
$$\rho a(X) = \alpha(\varphi(X)) + \theta(\nu(X)), \qquad \sigma a(X) = \nu(\varphi X) + \delta(\alpha(X))$$
$$\alpha(X)\alpha(Y) + \nu(X)\nu(Y) = 0.$$

The above equations are consistent iff

$$\eta(b_\star X)ob = a(X), \quad \alpha(X) = \nu(X) = 0, \quad \rho = \sigma = 0. \tag{2.6}$$

Substituting these in (2.3) (a), (b) we obtain

$$(a) \quad \varphi(b_\star X) = b_\star \varphi X \qquad (b) \quad \xi = b_\star \xi. \tag{2.7}$$

Thus we have

Theorem 2.1. *The necessary and sufficient conditions for a submanifold of V_{2m+1} to be a Lorentzian Para-contact submanifold are* (2.6) *and* (2.7).

Theorem 2.2. *Let us denote the Nijenhuis tensors in V_{2m+1} and V_{2m-1} by N and n, determined by φ and ϕ respectively, then $N(b_\star X, b_\star Y)= b_\star n(X,Y)$.*

Proof. In consequence of (2.2) and (2.7) (a) we have

$$\begin{aligned}\varphi(\varphi[b_\star X, b_\star Y]) &= \varphi(\varphi(\nabla_{b_\star X} b_\star Y - \nabla_{b_\star Y} b_\star X))\\ &= b_\star\phi^2(D_X Y) - b_\star\phi^2(D_Y X) = b_\star\phi^2([X,Y])\end{aligned}$$

hence

$$N(b_\star X, b_\star Y) = b_\star n(X,Y).$$

Definition 2.1. An Lorentzian Para-contact manifold is said to be normal if

$$N(X,Y) + d\eta(X,Y)\xi = 0.$$

Theorem 2.3. *If V_{2m+1} is normal then V_{2m-1} is also normal.*

Proof. We have from Thorem (2.2)

$$N(b_\star X, b_\star Y) = b_\star n(X,Y)$$

Now $$N(b_\star X, b_\star Y) + ((\nabla_{b_\star X}\eta)(b_\star Y) - (\nabla_{b_\star Y}\eta)(b_\star X))\xi = 0$$

$$n(X,Y) + ((D_X a)(Y) - (D_Y a)(X))\xi - 0.$$

This shows that if V_{2m+1} is normal then V_{2m-1} is also normal.

Theorem 2.4. *When V_{2m-1} is an Lorentzian Para-contact submanifold in a Lorentzian Para-contact manifold we have*

$$\begin{aligned}&(i)\quad \eta(M) = 0, \qquad \eta(N) = 0,\\ (2.8)\qquad &(ii)\quad \varphi(M) = \delta N, \qquad \varphi N = \theta M\\ &(iii)\quad \delta = \theta \quad \text{and} \quad \delta\theta = 1.\end{aligned}$$

Proof. We have $g(\varphi M, b_\star X) - g(M, b_\star\varphi X) = 0$ which gives

$$g(-b_\star p + \delta N, b_\star X) - g(M, b_\star\varphi X) = 0.$$

Using (2.1) (a), (b), (c), (d) we get

$$g(p, X) = 0 \implies p = 0.$$

Similarly we can get $q = 0$, putting this value in (2.3) (c) and (d) we get

$$\varphi(M) = \delta N, \quad \varphi(N) = \theta M$$

Pre-multiplying (2.3) (c), (d) by φ and using (1.1) and equating tangential and normal parts we have

$$\eta(M) = 0, \quad \eta(N) = 0, \quad \delta\theta = 1.$$

We also have
$$g(\varphi M, N) - g(M, \varphi N) = 0.$$
Putting the values of M and N as above we get
$$\delta = \theta.$$

Theorem 2.5. *Let V_{2m+1} be an Lorentzian Para-contact cosymplectic manifold then V_{2m-1} is also an LP–cosymplectic manifold and*

$$H(X, \varphi Y) = \delta K(X, Y), \quad K(X, \varphi Y) = \delta H(X, Y) \tag{2.9}$$

where H and K are symmetric bilinear functions in V_{2m-1} and $\delta^2 = 1$.

Proof. We have
$$\begin{aligned}(\nabla_{b_\star X}\eta)(b_\star Y) = 0 &\implies X(a(Y) - a(D_X Y) = 0\\ &\implies (D_X a)(Y) = 0.\end{aligned}$$
Also
$$(\nabla_{b_\star X}\varphi)(b_\star Y) = 0 \implies \nabla_{b_\star X} b_\star Y = \varphi(\nabla_{b_\star X} b_\star Y)$$
which gives
$$\begin{aligned}&b_\star D_X \varphi Y + H(X, \varphi Y)M + K(X, \varphi Y)N\\ &= b_\star \varphi(D_X Y) + H(X, Y)\delta N + K(X, Y)\delta M.\end{aligned}$$
This equation implies that $(D_X\varphi)(Y) = 0$ and (2.9) are satisfied. This completes the proof.

Theorem 2.6. *Let V_{2m+1} be an LP–nearly cosymplectic manifold. Then V_{2m-1} is also an LP–nearly cosymplectic manifold and*
$$H(X, \varphi Y) + H(Y, \varphi X) - 2\delta H(X, Y) = 0$$
and
$$K(X, \varphi Y) + K(Y, \varphi X) - 2\delta K(X, Y) = 0$$
where $\delta^2 = 1$.

Proof. For an LP–nearly cosymplectic manifold we have
$$(\nabla_{b_\star X}\varphi)(b_\star Y) + (\nabla_{b_\star Y}\varphi)(b_\star X) = 0$$
$$\nabla_{b_\star X} b_\star \varphi Y + \nabla_{b_\star Y} b_\star \varphi X = \varphi(\nabla_{b_\star X} b_\star Y) + \varphi(\nabla_{b_\star Y} b_\star X)$$
or
$$\begin{aligned}&b_\star D_X \varphi Y + H(X, \varphi Y)M + K(X, \varphi Y)N + b_\star D_Y \varphi X\\ &+ H(Y, \varphi X)M + K(Y, \varphi X)N = b_\star \varphi(D_X Y) + b_\star \varphi(D_Y X)\\ &+ H(X, Y)\delta N + H(Y, X)\delta N + K(X, Y)\delta M + K(Y, X)\delta M.\end{aligned}$$

This equation implies

$$(D_X\varphi)(Y) + (D_Y\varphi)(X) = 0$$

and

$$H(X, \varphi Y) + H(Y, \varphi X) - 2\delta K(X, Y) = 0$$

$$K(X, \varphi Y) + K(Y, \varphi X) - 2\delta H(X, Y) = 0.$$

Theorem 2.7. *Let V_{2m-1} be a submanifold tangent to the sturcture vector field ξ of an Lorentzian Para-Sasakian manifold V_{2m+1}. If V_{2m-1} is totally umbilical then V_{2m-1} is totally geodesic.*

Proof. From Gauss' equation we have

$$\nabla_{b_\star X}\xi = b_\star D_X\xi + H(X, \xi)M + K(X, \xi)N,$$

or

$$b_\star \varphi X = b_\star D_X\xi + H(X, \xi)M + K(X, \xi)N.$$

Equating tangential and normal parts we get

$$\varphi X = D_X\xi \qquad \text{and} \qquad H(X, \xi) = 0, \quad K(X, \xi) = 0.$$

Thus

$$H(\xi, \xi) = 0, \quad K(\xi, \xi) = 0.$$

If V_{2m-1} is totally umbilical, then $H(X, Y) = \mu g(X, Y) = K(X, Y)$. Writing ξ for both X and Y we get

$$H(\xi, \xi) = K(\xi, \xi) = 0 \implies g(\xi, \xi) = 0 \implies \mu = 0$$

which implies that

$$H(X, Y) = K(X, Y) = 0.$$

Thus V_{2m-1} is totally geodesic.

If V_{2m-1} is totally geodesic then $H(X, \xi) = 0$ that is φX is tangent to V_{2m-1} and hence V_{2m-1} is an invariant submanifold.

Theorem 2.8. *Let V_{2m-1} be a submanifold of a Lorentzian Para-Sasakian manifold. V_{2m+1} is tangent to the structure vector field ξ of V_{2m+1}. Then vector field ξ is parallel with respect to the induced connection on V_{2m-1} if and only if V_{2m-1} is an anti-invariant submanifold in V_{2m+1}.*

Proof. We have for the tangent ξ of V_{2m-1}

$$\nabla_{b_\star X}\xi = b_\star \varphi X = b_\star D_X\xi + H(X, \xi)M + K(X, \xi)N. \tag{2.10}$$

Since ξ is parallel with respect to the induced connection we have

$$D_X\xi = 0$$

From (2.10) we have

$$\varphi X = H(X,\xi)M + K(X,\xi)N$$

Hence φX is normal to V_{2m-1}. Thus $\varphi X \in T_X(V_{2m-1})^{\perp}$ for every vector field X on V_{2m-1}. Thus V_{2m-1} is anti-invariant. Conversely if V_{2m-1} is anti-invariant then $\varphi X = H(X,\xi)M + K(X,\xi)N$, hence $D_X\xi = 0$. This completes the proof.

Theorem 2.9. *Let V_{2m-1} be a submanifold of an Lorentzian Para-Sasakian manifold of V_{2m+1}. If the structure vector field ξ is normal to V_{2m-1}, then V_{2m-1} is totally geodesic if and only if V_{2m-1} is an anti-invariant submanifold.*

Proof. Since ξ is normal to V_{2m-1} we have

$$\begin{aligned} g(b_\star\varphi X, b_\star Y) = \; & g(b_\star\nabla_X\xi, b_\star Y) = g(-b_\star{}'H(X), b_\star Y) + \; g(l(X)\xi, b_\star Y) \\ & = g(-b_\star{}'K(X), b_\star Y) - g(l'(X)\xi, b_\star Y), \end{aligned}$$

or

$$g(b_\star\varphi X, b_\star Y) = -g('H(X),Y) = g('K(X),Y) \text{ for any } X$$

and Y on V_{2m-1}. Hence Φ, H and K are symmetric, hence $g(b_\star\varphi X, b_\star Y) = g('H(X),Y) = 0 = g('K(X),Y)$. If V_{2m-1} is totally geodesic then

$$'K(X) = {}'H(X) = 0 \implies \varphi(X) \in T_X(V_{2m-1}).$$

Hence V_{2m-1} is anti-invariant.
Conversely if V_{2m-1} is anti-invariant then

$$g('H(X),Y) = 0 = g('K(X),Y)$$
$$'H(X) = 0 = {}'K(X) \qquad H(X,Y) = 0 = K(X,Y)$$

hence V_{2m-1} is totally geodesic.

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