*Publ. Math. Debrecen*
**44 / 3-4** (1994), 243–248

# On a problem of D. Brydak

By M. Czerni (Kraków)

## 1. Introduction

At the Third International Symposium on functional equations and inequalities at Noszvaj (Hungary) in September 1986, D. Brydak put the following slighty reformulated problem (see supplement in [2] p. 36):

Let $f : J \to J$, where $J = [0, \alpha)$, $\alpha > 0$ be strictly increasing and continuous in $J$. Moreover, let $0 < f(x) < x$ for $x \in (0, \alpha)$. Let $g : J \to \mathbb{R}_+$ be a continuous in $J$. Let the equation

$$\varphi[f(x)] = g(x)\varphi(x), \quad x \in J, \tag{1}$$

have a continuous solution, positive in $(0, \alpha)$ and depending on an arbitrary function. Let $\psi : J \to \mathbb{R}_+$ be a continuous solution of the inequality

$$\psi[f(x)] \le g(x)\psi(x), \quad x \in J. \tag{2}$$

Does there always exist a solution $\varphi : J \to \mathbb{R}$ of equation (1) such that the limit

$$\lim_{x \to 0^+} \frac{\psi(x)}{\varphi(x)} \tag{3}$$

exists?

The answer to the above question is negative. In the present paper we are going to characterize such continuous, nonnegative solutions $\psi$ of (2) that for every solution $\varphi$ of (1) vanishing at zero only, the limit (3) does not exist.

At first we formulate assumptions about the given functions $f$ and $g$ as follows:

($\mathbf{H}_1$) Let $f : J \to J$ be strictly increasing and continuous in an interval $J = [0, \alpha)$. Moreover

$$0 < f(x) < x \quad \text{for} \quad x \in (0, \alpha). \tag{4}$$

1980 *Mathematics Subject Classification* (1985 *Revision*): Primary 39 C 05.

($\mathbf{H}_2$) Let $g : J \to \mathbb{R}$ be continuous in the interval $J$ and $g(x) > 0$ for $x \in (0, \alpha)$.

In the sequel we shall consider the following classes of functions:

*Definition 1.* We denote by $\Psi$ the family of all continuous, nonnegative solutions $\psi : J \to \mathbb{R}$ of the inequality (2) satisfying the condition

$$\psi(0) = 0.$$

We denote by $\Phi$ the family of all solutions $\varphi : J \to \mathbb{R}$ of equation (1) satisfying the conditions

$$\varphi(x) \neq 0 \quad \text{for} \quad x \in (0, \alpha), \quad \varphi(0) = 0.$$

It is necessary for further considerations to have the condition $\Phi \neq \emptyset$ fulfilled. For this reason (see [1]) we shall assume

($\mathbf{H}_3$) The sequence $\{G_n\}_{n \in \mathbb{N}}$ given by

$$G_n(x) = \prod_{i=0}^{n-1} g[f^i(x)] \quad \text{for } x \in J,\ n \in \mathbb{N} \tag{5}$$

where $f^i$ is the $i$-th iterate of the function $f$, i.e. $f^0 = Id$, $f^{n+1} = f \circ f^n$ converges to zero almost uniformly in the interval $J$.

*Remark.* If hypothesis ($\mathbf{H}_3$) is fulfilled, then equation (1) has a continuous solution in $J$ depending on an arbitrary function and every continuous solution $\varphi$ satisfies the condition $\varphi(0) = 0$ (see [4] p. 48). In particular ($\mathbf{H}_3$) implies that equation (1) has a continuous solution, positive in $(0, \alpha)$.

Finally, we introduce the following subclass of $\Phi$:

*Definition 2.* Let $\psi \in \Psi$ and $a \in \mathbb{R}$. We denote by $\Phi_a^\psi$ the family of all functions $\varphi \in \Phi$ such that the limit (3) exists and is equal to $a$.

Thus we may reformulate Brydak's problem as follows:
Is the formula

$$\bigwedge_{\psi \in \Psi} \bigvee_{a \in \mathbb{R}} \Phi_a^\psi \neq \emptyset$$

true?

## 2. Results

The following theorem contain results which are proved in [1] and will be needed in the sequel.

**Theorem 1.** *Let the hypotheses* $(\mathbf{H}_1)$ – $(\mathbf{H}_3)$ *be fulfilled and let* $\psi \in \Psi$*. Then there exists the limit*

$$\psi_0(x) := \lim_{n\to\infty} \frac{\psi[f^n(x)]}{G_n(x)} \quad \text{for} \quad x \in (0,\alpha),$$

*where* $\{G_n\}$ *is defined by formula* (5) *and the function*

$$\varphi_0(x) = \begin{cases} \psi_0(x) & \text{for} \quad x \in (0,\alpha) \\ 0 & \text{for} \quad x = 0 \end{cases} \tag{6}$$

*is a solution of equation* (1) *in* $J$*, upper semicontinuous in* $J$*, continuous at zero and fulfilling the inequalities*

$$0 \le \varphi_0(x) \le \psi(x). \tag{7}$$

Introduce the notation

$$\gamma_n(x) = \frac{\psi[f^n(x)]}{\varphi[f^n(x)]} \quad x \in J,\ n \in \mathbb{N}.$$

Now, we formulate the following

**Theorem 2.** *Let the hypotheses* $(\mathbf{H}_1)$ – $(\mathbf{H}_3)$ *be fulfilled and let* $\psi \in \Psi$ *and* $a \in \mathbb{R}$*. If* $\varphi \in \Phi_a^\psi$*, then for every* $x_0 \in (0,\alpha)$ *the sequence* $(\gamma_n)$ *converges to* $a$*, uniformly in the interval* $(0, x_0]$*. Moreover*

$$a\varphi(x) = \varphi_0(x) \quad \text{for} \quad x \in J \tag{8}$$

*where* $\varphi_0$ *is given by formula* (6).

Proof. Let us fix an $x_0 \in J \setminus \{0\}$ and suppose that $(\gamma_n)$ does not converge to $a$, uniformly in $(0, x_0]$, i.e.

$$\bigvee_{\varepsilon>0} \bigwedge_{n\in\mathbb{N}} \bigvee_{k_n \ge n} \bigvee_{x_n \in (0,x_0]} |\gamma_{k_n}(x_n) - a| \ge \varepsilon > 0. \tag{9}$$

Without loss of generality we may assume that the sequence $\{k_n\}$ is strictly increasing. But from hypothesis $(\mathbf{H}_1)$ we obtain

$$0 < f^{k_n}(x_n) \le f^{k_n}(x_0)$$

and this implies that $\lim_{n\to\infty} f^{k_n}(x_n) = 0$, by virtue of (4). Thus the estimation in (9) proves that $\lim_{x\to 0} \frac{\psi(x)}{\varphi(x)}$ is not equal to $a$, contrary to our assumption on $\varphi$. This ends the proof of the first part of the theorem.

We have also

$$\varphi_0(x) = \psi_0(x) = \lim_{n\to\infty} \frac{\psi[f^n(x)]\varphi(x)}{G_n(x)\varphi(x)} = \\ = \left(\lim_{n\to\infty} \gamma_n(x)\right) \varphi(x) = a\varphi(x) \quad \text{for} \quad x \ne 0$$

and

$$\varphi_0(0) = 0 = a\varphi(0).$$

Thus we obtain (8).

**Theorem 3.** *Let the hypotheses* $(\mathbf{H}_1)$ – $(\mathbf{H}_3)$ *be fulfilled and let* $\psi \in \Psi$, $\varphi \in \Phi$ *and* $a \in \mathbb{R}$. *If there exists an* $x_0 \in J \setminus \{0\}$ *such that the sequence* $(\gamma_n)$ *converges to* $a$, *uniformly in the interval* $[f(x_0), x_0]$, *then* $\varphi \in \Phi_a^\psi$.

Proof. It is sufficient to show that the limit (3) exists. Let us fix an $\varepsilon > 0$. Thus there exists a positive integer $N$ that for every $n > N$ and $x \in [f(x_0), x_0]$ we have

$$|\gamma_n(x) - a| < \varepsilon. \tag{10}$$

Let us put $\delta := f^{N+1}(x_0)$. If we take $t \in (0, \delta)$, then $t = f^n(x)$ for some $n > N$ and $x \in [f(x_0), x_0]$ (see [4] p.21). Thus we obtain

$$\left| \frac{\psi(t)}{\phi(t)} - a \right| < \varepsilon$$

by virtue of (10) and this ends the proof of the theorem.

Now, we are going to give some simple result concerning the case where $\phi_0(x)$ is identically equal to zero.

**Theorem 4.** *Let the hypotheses* $(\mathbf{H}_1)$ – $(\mathbf{H}_3)$ *be fulfilled and let* $\psi \in \Psi$ *be such that* $\varphi_0(x) = 0$ *for* $x \in J$. *If* $\varphi \in \Phi$ *fulfils the condition*

$$\bigvee_{x_0 \in J \setminus \{0\}} \ \bigvee_{m > 0} \ \bigwedge_{x \in [f(x_0), x_0]} |\varphi(x)| > m,$$

*then* $\varphi \in \Phi_0^\psi$.

Proof. Since the sequence $\left(\frac{\psi[f^n(x)]}{G_n(x)}\right)$ is decreasing for every $x \in J$ (see [1]), in view of the Dini's theorem $\lim_{n\to\infty} \frac{\psi[f^n(x)]}{G_n(x)} = 0$, uniformly in $[f(x_0), x_0]$. Consequently for every $\varepsilon > 0$ there exists such a positive integer $N$ that for $n > N$ and $x \in [f(x_0), x_0]$ we have

$$|\gamma_n(x)| = \left| \frac{\psi[f^n(x)]}{G_n(x)\varphi(x)} \right| \le \frac{1}{m} \left( \frac{\psi[f^n(x)]}{G_n(x)} \right) < \varepsilon$$

and by virtue of Theorem 3 this ends the proof of the theorem.

Finally, we shall prove

**Theorem 5.** *Let the hypotheses* $(\mathbf{H}_1)$ – $(\mathbf{H}_3)$ *be fulfilled,* $\psi \in \Psi$ *and* $a \in \mathbb{R} \setminus \{0\}$. *If* $\varphi \in \Phi_a^\psi$, *then the solution* $\phi_0$ *defined by* (6) *is continuous in* $J$.

Proof. Since the limit (3) exists and is not equal to zero then $\psi(x) \neq 0$ for every $x$ in a vicinity of zero. Thus, and because of $(\mathbf{H}_1)$ there exists an $x_0 \in J \setminus \{0\}$ such that

$$M := |a|^{-1} \max_{x \in [f(x_0), x_0]} \psi(x) > 0.$$

Let us fix an $\varepsilon > 0$. By virtue of Theorem 2, for $\frac{\varepsilon}{M}$ there exists such an $N > 0$ that for $n > N$ and $x \in (0, x_0]$ we have

$$|\gamma_n(x) - a| < \frac{\varepsilon}{M}. \tag{11}$$

Hence, in view of (8) and (11) we obtain

$$\left| \frac{\psi[f^n(x)]}{G_n(x)} - \varphi_0(x) \right| = \left| \frac{\psi[f^n(x)]\varphi(x)}{G_n(x)\varphi(x)} - \varphi_0(x) \right| = |\gamma_n(x)\varphi(x) - \varphi_0(x)| =$$
$$= \frac{\varphi_0(x)}{|a|} |\gamma_n(x) - a| \leq M \frac{\varepsilon}{M} = \varepsilon \quad \text{for} \quad x \neq 0.$$

Moreover, let us note that the estimation

$$0 \leq \frac{\psi[f^n(x)]}{G_n(x)} \leq \psi(x)$$

implies that for every $n \in \mathbb{N}$

$$\lim_{x \to 0} \frac{\psi[f^n(x)]}{G_n(x)} = 0.$$

Thus the decreasing sequence

$$\eta_n(x) := \begin{cases} \frac{\psi[f^n(x)]}{G_n(x)} & \text{for} \quad x \in (0, x_0] \\ 0 & \text{for} \quad x = 0 \end{cases}$$

of continuous functions tends uniformly to $\varphi_0$ on $[0, x_0]$ and this proves that $\varphi_0$ is continuous in $[0, x_0]$. Consequently (see [4] p. 70) $\varphi_0$ is continuous in $J$. This ends the proof of the theorem.

## 3. Concluding remarks

Let $\psi \in \Psi$. There are four possible cases.

*Case 1:* $\varphi_0$ is continuous and $\varphi_0 > 0$ on $J \setminus \{0\}$.

In this case $\varphi_0 \in \Phi_1^{\psi}$. (see [1], Theorem 3.9).

*Case 2:* $\varphi_0(x) = 0$ for $x \in J$.

In this case $\Phi_0^{\psi} \neq \emptyset$, by virtue of Theorem 4. Indeed, it is sufficient to take an arbitrary function $\bar{\varphi}$ defined in $[f(x_0), x_0]$ and fulfilling the conditions

$$\bar{\varphi}[f(x_0) = g(x_0)\bar{\varphi}(x_0),$$
$$|\bar{\varphi}(x)| > m, \quad x \in [f(x_0), x_0].$$

Thus we can construct its extension $\varphi$, using equation (1), successively in the intervals $[f^{k+1}(x_0),\, f^k(x_0)]$ for any integer values of $k$. (see [4] p. 32). If we put additionally $\varphi(0) = 0$ then $\varphi \in \Phi_0^{\psi}$.

*Case 3:* $\varphi_0$ is continuous, it is not identically equal to zero but there exists such an $\bar{x} \in J \setminus \{0\}$ that $\varphi_0(\bar{x}) = 0$.

In this case $\Phi_a^{\psi} = \emptyset$ for every $a \in \mathbb{R}$, by virtue of (8).

*Case 4:* $\varphi_0$ is not continuous.

In this case $\Phi_a^{\psi} = \emptyset$ for every $a \in \mathbb{R}$, by virtue of Theorem 5. An example of such a function $\psi \in \Psi$ that $\phi_0$ is not continuous is found in [3].

Consequently, the answer to Brydak's problem is negative (case 3 and 4).

## References

[1] D. Brydak, On functional inequalities in a single variable, Dissertationes Math. **CLX**, 1979.

[2] D. Brydak, Problem 2, Report of the Third International Symposium on Functional Equations and Inequalities, Noszvaj 1986, *Publ. Math. Debrecen* **38** (1991), 36.

[3] D. Brydak and B. Choczewski, On non-homogeneous linear functional inequality, *Zeszyty Naukowe U.J. Prace Mat.* **18** (1976), 89–92.

[4] M. Kuczma, Functional equations in a single variable, *Polish Scientific Publishers*, *Warszawa*, 1986.

MAREK CZERNI
DEPT. OF MATHEMATICS
PEDAGOGICAL UNIVERSITY OF CRACOW
PODCHORĄŻYCH 2
PL–30–084 KRAKÓW
POLAND

*(Received July 27, 1992; revised November 29, 1993)*