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On weakly symmetric and weakly Ricci symmetric warped product manifolds

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Abstract. This paper is concerned with some results on weakly symmetric and weakly Ricci symmetric warped product manifolds. We prove the necessary and sufficient condition for a warped product manifold to be weakly symmetric and weakly Ricci symmetric. On the basis of these results two proper examples of warped product weakly symmetric and weakly Ricci symmetric manifolds are presented.

1. Introduction

Let M, $\dim M = n \geq 3$, be a semi-Riemannian manifold with Levi–Civita connection ∇ and semi-Riemannian metric g. Let R, S and κ be the curvature tensor of type $(0,4)$, Ricci tensor of type $(0,2)$ and scalar curvature of (M,g) respectively. The manifold M is locally symmetric if $\nabla R = 0$, which is equivalent to the fact that for each point $x \in M$, the local geodesic symmetry is an isometry. For 2-dimensional manifolds being of locally symmetric and being of constant curvature are equivalent. But for $n \geq 3$, the locally symmetric manifolds are a generalization of the manifolds of constant curvature. A full classification of locally symmetric manifolds is given by Cartan [3] for Riemannian case and Cahen and Parker ([4], [5]) for indefinite case. The semi-Riemannian manifold M is said to be Ricci-symmetric if $\nabla S = 0$. Every locally symmetric manifold is Ricci-symmetric but not conversely, in general. However, the converse is true for dimension 3.

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During the last six decades the notion of locally symmetric manifolds has been weakened by many authors in different directions such as recurrent manifolds by Walker [50], generalized recurrent manifolds by Dubey [20], quasi-generalized recurrent manifolds by Shaikh and Roy [43], weakly generalized recurrent manifolds by Shaikh and Roy [44], hyper-generalized recurrent manifolds by Shaikh and Patra [42], semi-symmetric manifolds by Szabó [46], pseudosymmetric manifolds by Chaki [6], pseudosymmetric manifolds by Deszcz [19], weakly symmetric manifolds by Selberg [31] and weakly symmetric manifolds by Tamássy and Binh [47]. We note that the notion of pseudosymmetry by Deszcz [19] is different to that by Chaki [6]. Also, the notion of weakly symmetric manifolds by Selberg [31] is different to that by Tamássy and Binh [47] and throughout the paper we will confined ourselves with the notion of weakly symmetric manifolds by Tamássy and Binh [47].

Let $U = \{x \in M : \nabla R \neq 0 \text{ and } R \neq 0 \text{ at } x\}$. Then (M, g) is said to be recurrent [50] if on $U \subset M$, we have $\nabla R = H \otimes R$, where H is an unique 1-form. It is obvious that the 1-form H is non-zero at every point of U. Such a manifold is denoted by K_n.

Let $U_R = \{x \in M : R \neq 0 \text{ and } \nabla R - A \otimes R \neq 0 \text{ at } x \text{ for all 1-forms } A\}$. Then (M, g) is said to be weakly symmetric [47] if on $U_R \subset M$, we have $\nabla R = L$, where L is a tensor of type (0,5) defined by

$$\begin{aligned} L(X, X_1, X_2, X_3, X_4) = {} & A(X)R(X_1, X_2, X_3, X_4) + B(X_1)R(X, X_2, X_3, X_4) \\ & + C(X_2)R(X_1, X, X_3, X_4)r + D(X_3)R(X_1, X_2, X, X_4) \\ & + E(X_4)R(X_1, X_2, X_3, X), \end{aligned} \tag{1.1}$$

for all vector fields X, $X_i \in \chi(M)$ $(i = 1, 2, 3, 4)$, $\chi(M)$ being the Lie algebra of all smooth vector fields on M, where A, B, C, D, E are 1-forms on M. It is clear that the 1-forms A, B, C, D, E can not be zero at every point of U_R. Such a manifold is denoted by WS_n. From the definition it follows that every K_n is a WS_n but not conversely. The existence of a WS_n is proved by Prvanović [29]. We note that all 1-forms A, B, C, D, E in (1.1) are not distinct. In fact, in a WS_n the 1-form $B = C$ and $D = E$ [11]. Hence the reduced defining condition of a WS_n is given by

$$\begin{aligned} (\nabla_X R)(X_1, X_2, X_3, X_4) = {} & A(X)R(X_1, X_2, X_3, X_4) + B(X_1)R(X, X_2, X_3, X_4) \\ & + B(X_2)R(X_1, X, X_3, X_4) + D(X_3)R(X_1, X_2, X, X_4) \\ & + D(X_4)R(X_1, X_2, X_3, X). \end{aligned} \tag{1.2}$$

Especially, if (i) $B = D = \frac{1}{2}A$, (ii) $B = D \neq A$, then a WS_n turns into a pseudosymmetric manifold in the sense of Chaki [6] and extended recurrent manifold by Prvanović [30] respectively. If in (1.2), the 1-form A is replaced by $2A$ and D is replaced by A, then a WS_n turns into a generalized pseudosymmetric manifold by Chaki [8]. Again Prvanović [29] proved that if a WS_n is not pseudosymmetric in the sense of Chaki, then it is a B-space by Venzi [49]. For a WS_n we refer the survey work by De [10] and also references therein. Decomposable WS_n is studied by Binh [1]. The non-trivial examples of WS_n and decomposable WS_n are given by Shaikh and Jana [38]. Also Shaikh and his coauthors studied weakly symmetric manifolds with various generalized curvature tensors ([23], [32], [33], [34], [35], [36], [37], [40], [41], [45]). Weakly symmetric contact structure is also studied by De, Shaikh and others ([12], [18]).

Let $V = \{x \in M : S \neq 0 \text{ and } \nabla S \neq 0 \text{ at } x\}$. Then (M, g) is said to be Ricci recurrent [28] if on $V \subset M$, we have $\nabla S = W \otimes S$, where W is an unique 1-form. Every recurrent manifold is Ricci recurrent but not conversely.

Let $V_S = \{x \in M : S \neq 0 \text{ and } \nabla S - I \otimes S \neq 0 \text{ at } x \text{ for all 1-forms } I\}$. Then (M, g) is said to be weakly Ricci symmetric [48] if on $V_S \subset M$, we have $\nabla S = N$, where N is a tensor of type $(0, 3)$ defined by

$$N(X, X_1, X_2) = I(X)S(X_1, X_2) + J(X_1)S(X, X_2) + K(X_2)S(X_1, X), \quad (1.3)$$

for all vector fields X, X_1, $X_2 \in \chi(M)$, where I, J, K are three 1-forms. It is clear that the 1-forms I, J, K can not be zero at every point of V_S. Such a manifold is denoted by WRS_n. Especially, if $J = K = \frac{1}{2}I$, then a WRS_n turns into a pseudo Ricci symmetric manifold in the sense of Chaki [7]. Also, if the 1-form I is replaced by $2I$, then a WRS_n reduces to a generalized pseudo Ricci symmetric manifold by Chaki and Koley [9]. Hence the defining condition of a WRS_n is given by,

$$(\nabla_X S)(X_1, X_2) = I(X)S(X_1, X_2) + J(X_1)S(X, X_2) + K(X_2)S(X_1, X). \quad (1.4)$$

The existence of WRS_n is ensured by Shaikh and Jana by several examples [39]. WRS_n is also studied by De and his coauthors ([13], [14], [15], [16]). Again in [25] Mantica and Molinari proved that if the Ricci tensor is non-singular, then $J = K$, and they also obtained a necessary and sufficient condition for the 1-form I to be closed. Recently, Mantica and Molinari [26] studied WZS_n, where Z is a generalized symmetric tensor of type $(0, 2)$.

The object of the present paper is to study WS_n and WRS_n warped product manifolds. We obtain the necessary and sufficient condition for a warped product manifold to be WS_n and WRS_n. As a particular case of our results (see, Theorem 3.1 and 4.1), we can obtain the result of Binh [1] and also the results of [17]. Basing on these results two proper examples of warped product WS_n and WRS_n are presented.

2. Warped product manifolds

The study of warped product manifolds was initiated by Kručkovič [24] in 1957. Again in 1969 Bishop and O'Neill [2] also obtained the same notion of the warped product manifolds while they were constructing a large class of complete manifolds of negative curvature. Warped product manifolds are generalizations of the Cartesian product of semi-Riemannian manifolds. Let $(\bar{M}, \bar{g})$ and $(\tilde{M}, \tilde{g})$ be two semi-Riemannian manifolds of dimension p and $(n-p)$ respectively $(1 \leq p < n)$, and f is a positive smooth function on $\bar{M}$. Let $\bar{M}$ and $\tilde{M}$ be covered with coordinate charts $(U; x^1, x^2, \dots, x^p)$ and $(V; y^1, y^2, \dots, y^{n-p})$ respectively. Then the warped product $M = \bar{M} \times_f \tilde{M}$ is the product manifold $\bar{M} \times \tilde{M}$ of dimension n furnished with the metric $g = \pi^*(\bar{g}) + (f \circ \pi)\sigma^*(\tilde{g})$, where $\pi : M \to \bar{M}$ and $\sigma : M \to \tilde{M}$ are natural projections such that $M = \bar{M} \times \tilde{M}$ is covered with the coordinate chart $(U \times V; x^1, x^2, \dots, x^p, x^{p+1} = y^1, x^{p+2} = y^2, \dots, x^n = y^{n-p})$. Then the local components of the metric g with respect to this coordinate chart are given by

$$g_{ij} = \begin{cases} \bar{g}_{ij} & \text{for } \ i = a \text{ and } j = b, \\ f\tilde{g}_{ij} & \text{for } i = \alpha \text{ and } j = \beta, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Here $a, b \in \{1, 2, \dots, p\}$ and $\alpha, \beta \in \{p+1, p+2, \dots, n\}$. We note that throughout the paper we consider $a, b, c, \dots \in \{1, 2, \dots, p\}$ and $\alpha, \beta, \gamma, \dots \in \{p+1, p+2, \dots, n\}$ and $i, j, k, \dots \in \{1, 2, \dots, n\}$. Here $\bar{M}$ is called the base, $\tilde{M}$ is called the fiber and f is called warping function of the warped product $M = \bar{M} \times_f \tilde{M}$. We denote by Γ^i_{jk}, R_{ijkl}, S_{ij} and κ as the components of Levi–Civita connection ∇, the Riemann–Chirstoffel curvature tensor R, Ricci tensor S and the scalar curvature of (M, g) respectively. Moreover we consider that, when Ω is a quantity formed with respect to g, we denote by $\bar{\Omega}$ and $\tilde{\Omega}$, the similar quantities formed with respect to $\bar{g}$ and $\tilde{g}$ respectively.

Then the non-zero local components of Levi–Civita connection ∇ of (M, g)

are given by

$$\Gamma^a_{bc} = \bar{\Gamma}^a_{bc}, \quad \Gamma^\alpha_{\beta\gamma} = \tilde{\Gamma}^\alpha_{\beta\gamma}, \quad \Gamma^a_{\beta\gamma} = -\frac{1}{2}\bar{g}^{ab} f_b \tilde{g}_{\beta\gamma}, \quad \Gamma^\alpha_{a\beta} = \frac{1}{2f} f_a \delta^\alpha_\beta, \tag{2.2}$$

where $f_a = \partial_a f = \frac{\partial f}{\partial x^a}$. The local components $R_{hijk} = g_{hl} R^l_{ijk} = g_{hl}(\partial_k \Gamma^l_{ij} - \partial_j \Gamma^l_{ik} + \Gamma^m_{ij}\Gamma^l_{mk} - \Gamma^m_{ik}\Gamma^l_{mj})$, $\partial_k = \frac{\partial}{\partial x^k}$, of the Riemann–Christoffel curvature tensor R of (M, g) which may not vanish identically are the following:

$$R_{abcd} = \bar{R}_{abcd}, \quad R_{a\alpha b\beta} = f T_{ab} \tilde{g}_{\alpha\beta}, \quad R_{\alpha\beta\gamma\delta} = f \tilde{R}_{\alpha\beta\gamma\delta} - f^2 P \tilde{G}_{\alpha\beta\gamma\delta}, \tag{2.3}$$

where $G_{ijkl} = g_{il}g_{jk} - g_{ik}g_{jl}$ are the components of Gaussian curvature and

$$T_{ab} = -\frac{1}{2f}\left(\nabla_b f_a - \frac{1}{2f} f_a f_b\right), \quad \mathrm{tr}(T) = g^{ab} T_{ab},$$
$$Q = f((n-p-1)P - \mathrm{tr}(T)), \quad P = \frac{1}{4f^2} g^{ab} f_a f_b.$$

Again, the non-zero local components of the Ricci tensor $S_{jk} = g^{il} R_{ijkl}$ of (M, g) are given by

$$S_{ab} = \bar{S}_{ab} - (n-p) T_{ab}, \quad S_{\alpha\beta} = \tilde{S}_{\alpha\beta} + Q \tilde{g}_{\alpha\beta}. \tag{2.4}$$

The scalar curvature κ of (M, g) is given by

$$\kappa = \bar{\kappa} + \frac{\tilde{\kappa}}{f} - (n-p)[(n-p-1)P - 2\ \mathrm{tr}(T)]. \tag{2.5}$$

Again, the non-zero local components of ∇R and ∇S are given by [22]:

$$\begin{cases} \text{(i)} & \nabla_e R_{abcd} = \bar{\nabla}_e \bar{R}_{abcd}, \\ \text{(ii)} & \nabla_e R_{a\alpha b\beta} = f \bar{\nabla}_e T_{ab} \tilde{g}_{\alpha\beta}, \\ \text{(iii)} & \nabla_e R_{\alpha\beta\gamma\delta} = -f_e \tilde{R}_{\alpha\beta\gamma\delta} + f^2 P_e \tilde{G}_{\alpha\beta\gamma\delta}, \\ \text{(iv)} & \nabla_\epsilon R_{\alpha\beta\gamma\delta} = f \tilde{\nabla}_\epsilon \tilde{R}_{\alpha\beta\gamma\delta}, \\ \text{(v)} & \nabla_\epsilon R_{\alpha\beta\gamma d} = -\dfrac{f_d}{2} \tilde{R}_{\alpha\beta\gamma\epsilon} + \frac{f^2}{2} P_d \tilde{G}_{\alpha\beta\gamma\epsilon}, \\ \text{(vi)} & \nabla_\epsilon R_{abc\delta} = \dfrac{1}{2}\tilde{g}_{\epsilon\delta}(f_a T_{bc} - f_b T_{ac}) + \dfrac{1}{2} f^d R_{abcd} \tilde{g}_{\epsilon\delta},\ f^b = \bar{g}^{ab} fa, \end{cases} \tag{2.6}$$

$$\begin{cases} \text{(i)} & \nabla_e S_{ab} = \bar{\nabla}_e \bar{S}_{ab} - (n-p)\bar{\nabla}_e T_{ab}, \\ \text{(ii)} & \nabla_e S_{\alpha\beta} = Q_e \tilde{g}_{\alpha\beta} - \dfrac{f_e}{f}(\tilde{S}_{\alpha\beta} + Q\tilde{g}_{\alpha\beta}), \\ \text{(iii)} & \nabla_\epsilon S_{\alpha\beta} = \tilde{\nabla}_\epsilon \tilde{S}_{\alpha\beta}, \\ \text{(iv)} & \nabla_\epsilon S_{\alpha a} = -\dfrac{1}{2f}\tilde{S}_{\alpha\epsilon} f_a + \dfrac{1}{2}\tilde{g}_{\alpha\epsilon}\left[f^c \left(\bar{S}_{ca} - (n-p)T_{ca}\right) - \dfrac{Q}{f} f_a \right]. \end{cases} \tag{2.7}$$

For more detail informations about warped product we refer the reader to see [27].

3. Weakly symmetric warped product manifolds

Theorem 3.1. *Let $M = \bar{M} \times_f \tilde{M}$ be a non-flat warped product manifold. Then M is a WS_n*

$$\text{(i.e. } \nabla_l R_{hijk} = A_l R_{hijk} + B_h R_{lijk} + B_i R_{hljk} + D_j R_{hilk} + D_k R_{hijl}) \tag{3.1}$$

if and only if the following conditions hold:

(1) *base $\bar{M}$ is WS_p (i.e. $\bar{\nabla}_e \bar{R}_{abcd} = \bar{A}_e \bar{R}_{abcd} + \bar{B}_a \bar{R}_{ebcd} + \bar{B}_b \bar{R}_{aecd} + \bar{D}_c \bar{R}_{abed} + \bar{D}_d \bar{R}_{abce}$),*

(2) $\tilde{\nabla}_\epsilon \tilde{R}_{\alpha\beta\gamma\delta} - \big[\tilde{A}_\epsilon \tilde{R}_{\alpha\beta\gamma\delta} + \tilde{B}_\alpha \tilde{R}_{\epsilon\beta\gamma\delta} + \tilde{B}_\beta \tilde{R}_{\alpha\epsilon\gamma\delta} + \tilde{D}_\gamma \tilde{R}_{\alpha\beta\epsilon\delta} + \tilde{D}_\delta \tilde{R}_{\alpha\beta\gamma\epsilon}\big] = -fP\big[\tilde{A}_\epsilon \tilde{G}_{\alpha\beta\gamma\delta} + \tilde{B}_\alpha \tilde{G}_{\epsilon\beta\gamma\delta} + \tilde{B}_\beta \tilde{G}_{\alpha\epsilon\gamma\delta} + \tilde{D}_\gamma \tilde{G}_{\alpha\beta\epsilon\delta} + \tilde{D}_\delta \tilde{G}_{\alpha\beta\gamma\epsilon}\big]$,

(3) $\tilde{A}_\alpha \bar{R}_{abcd} = \tilde{B}_\alpha \bar{R}_{abcd} = \tilde{D}_\alpha \bar{R}_{abcd} = 0$,

(4) $\bar{\nabla}_e T_{ab} = \bar{A}_e T_{ab} + \bar{B}_a T_{eb} + \bar{D}_b T_{ae}$, $T_{ab}\big(\tilde{B}_\alpha \tilde{g}_{\beta\gamma} - \tilde{B}_\beta \tilde{g}_{\alpha\gamma}\big) = T_{ab}\big(\tilde{D}_\alpha \tilde{g}_{\beta\gamma} - \tilde{D}_\beta \tilde{g}_{\alpha\gamma}\big) = T_{ab}\big(\tilde{A}_\gamma \tilde{g}_{\alpha\beta} + \tilde{B}_\alpha \tilde{g}_{\gamma\beta} + \tilde{D}_\beta \tilde{g}_{\gamma\alpha}\big) = 0$,

(5) $f^d \bar{R}_{abcd} + (f_a + 2f\bar{B}_a)T_{bc} - (f_b + 2f\bar{B}_b)T_{ac} = 0$,
$f^d \bar{R}_{abcd} + (f_a + 2f\bar{D}_a)T_{bc} - (f_b + 2f\bar{D}_b)T_{ac} = 0$,

(6) $(f\bar{A}_e + f_e)\tilde{R}_{\alpha\beta\gamma\delta} = f^2(P_e - P\bar{A}_e)\tilde{G}_{\alpha\beta\gamma\delta}$,
$(2f\bar{B}_e + f_e)\tilde{R}_{\alpha\beta\gamma\delta} = f^2(P_e - 2P\bar{B}_e)\tilde{G}_{\alpha\beta\gamma\delta}$,
$(2f\bar{D}_e + f_e)\tilde{R}_{\alpha\beta\gamma\delta} = f^2(P_e - 2P\bar{D}_e)\tilde{G}_{\alpha\beta\gamma\delta}$,

where

$$A_i = \begin{cases} \bar{A}_i & \text{for } i = 1, \dots, p \\ \tilde{A}_i & \text{otherwise,} \end{cases} \qquad B_i = \begin{cases} \bar{B}_i & \text{for } i = 1, \dots, p \\ \tilde{B}_i & \text{otherwise,} \end{cases}$$

$$D_i = \begin{cases} \bar{D}_i & \text{for } i = 1, \dots, p \\ \tilde{D}_i & \text{otherwise.} \end{cases}$$

Proof. Let M be a non-flat weakly symmetric manifold. Then considering all possible cases of equation (3.1) for $h,i,j,k,l \in \{1,2,\dots,p\} \cup \{p+1,p+2,\dots,n\}$ and by putting their values from (2.3) and (2.6), we get our assertion easily. $\square$

As an immediate consequence of Theorem 3.1, we get the following results:

Corollary 3.1. *Let $M = \bar{M} \times_f \tilde{M}$ be a non-flat weakly symmetric warped product manifold such that*

$$\nabla_l R_{hijk} = A_l R_{hijk} + B_h R_{lijk} + B_i R_{hljk} + D_j R_{hilk} + D_k R_{hijl}.$$

Then

(1) *base is weakly symmetric,*

(2) *fiber is*

  (i) *of constant curvature if any one of*
  (a) $fA_e + f_e$ (b) $2fB_e + f_e$ (c) $2fD_e + f_e$ *is non-zero and*

  (ii) *weakly symmetric if*

$$P\big[\tilde{A}_\epsilon \tilde{G}_{\alpha\beta\gamma\delta} + \tilde{B}_\alpha \tilde{G}_{\epsilon\beta\gamma\delta} + \tilde{B}_\beta \tilde{G}_{\alpha\epsilon\gamma\delta} + \tilde{D}_\gamma \tilde{G}_{\alpha\beta\epsilon\delta} + \tilde{D}_\delta \tilde{G}_{\alpha\beta\gamma\epsilon}\big] = 0.$$

Remark. We note that if in a weakly symmetric warped product manifold the fiber is not of constant curvature, then $fA_e + f_e = 2fB_e + f_e = 2fD_e + f_e = 0$ and hence the manifold reduces to a pseudo symmetric manifold in the sense of Chaki.

Since the warped product is the generalization of a decomposable manifold, from Theorem 3.1 we get the following result of Binh [1].

Corollary 3.2 ([1]). *Let $M = \bar{M} \times \tilde{M}$ be a non-locally symmetric decomposable manifold. Then M is WS_n if and only if one of the decomposition is weakly symmetric and another is flat.*

Proof. A warped product manifold is decomposable if the warping function $f \equiv 1$. Then the conditions of Theorem 3.1 reduce to

$$\bar{\nabla}_e \bar{R}_{abcd} = \bar{A}_e \bar{R}_{abcd} + \bar{B}_a \bar{R}_{ebcd} + \bar{B}_b \bar{R}_{aecd} + \bar{D}_c \bar{R}_{abed} + \bar{D}_d \bar{R}_{abce},$$

$$\tilde{\nabla}_\epsilon \tilde{R}_{\alpha\beta\gamma\delta} = \tilde{A}_\epsilon \tilde{R}_{\alpha\beta\gamma\delta} + \tilde{B}_\alpha \tilde{R}_{\epsilon\beta\gamma\delta} + \tilde{B}_\beta \tilde{R}_{\alpha\epsilon\gamma\delta} + \tilde{D}_\gamma \tilde{R}_{\alpha\beta\epsilon\delta} + \tilde{D}_\delta \tilde{R}_{\alpha\beta\gamma\epsilon},$$

$$\tilde{A}_\alpha \bar{R}_{abcd} = \tilde{B}_\alpha \bar{R}_{abcd} = \tilde{D}_\alpha \bar{R}_{abcd} = 0, \quad \bar{A}_e \tilde{R}_{\alpha\beta\gamma\delta} = \bar{B}_e \tilde{R}_{\alpha\beta\gamma\delta} = \bar{D}_e \tilde{R}_{\alpha\beta\gamma\delta} = 0.$$

Now for a decomposable manifold we have $R_{abcd} = \bar{R}_{abcd}$, $R_{\alpha\beta\gamma\delta} = \tilde{R}_{\alpha\beta\gamma\delta}$. Then from the above reduced conditions the result follows. □

Since a pseudosymmetric manifold in the sense of Chaki [6] is a special case of a WS_n, Theorem 3.1 leads to the following:

Corollary 3.3. *Let $M = \bar{M} \times_f \tilde{M}$ be a non-flat warped product manifold. Then M is a pseudosymmetric manifold in the sense of Chaki (i.e. $\nabla_l R_{hijk} = 2A_l R_{hijk} + A_h R_{lijk} + A_i R_{hljk} + A_j R_{hilk} + A_k R_{hijl}$) if and only if all the following conditions hold:*

(1) *base $\bar{M}$ is pseudosymmetric (i.e. $\bar{\nabla}_e \bar{R}_{abcd} = 2\bar{A}_e \bar{R}_{abcd} + \bar{A}_a \bar{R}_{ebcd} + \bar{A}_b \bar{R}_{aecd} + \bar{A}_c \bar{R}_{abed} + \bar{A}_d \bar{R}_{abce}$),*

(2) $\tilde{\nabla}_\epsilon \tilde{R}_{\alpha\beta\gamma\delta} - \big[2\tilde{A}_\epsilon \tilde{R}_{\alpha\beta\gamma\delta} + \tilde{A}_\alpha \tilde{R}_{\epsilon\beta\gamma\delta} + \tilde{A}_\beta \tilde{R}_{\alpha\epsilon\gamma\delta} + \tilde{A}_\gamma \tilde{R}_{\alpha\beta\epsilon\delta} + \tilde{A}_\delta \tilde{R}_{\alpha\beta\gamma\epsilon}\big] = -fP\big[2\tilde{A}_\epsilon \tilde{G}_{\alpha\beta\gamma\delta} + \tilde{A}_\alpha \tilde{G}_{\epsilon\beta\gamma\delta} + \tilde{A}_\beta \tilde{G}_{\alpha\epsilon\gamma\delta} + \tilde{A}_\gamma \tilde{G}_{\alpha\beta\epsilon\delta} + \tilde{A}_\delta \tilde{G}_{\alpha\beta\gamma\epsilon}\big]$,

(3) $\tilde{A}_\alpha \bar{R}_{abcd} = 0$,

(4) $\bar{\nabla}_e T_{ab} = 2\bar{A}_e T_{ab} + \bar{A}_a T_{eb} + \bar{A}_b T_{ae}$ *and* $T_{ab}\tilde{A}_\alpha = 0$,

(5) $f^d \bar{R}_{abcd} + (f_a + 2f\bar{A}_a)T_{bc} - (f_b + 2f\bar{A}_b)T_{ac} = 0$,

(6) $(f\bar{A}_e + f_e)\tilde{R}_{\alpha\beta\gamma\delta} = f^2(P_e - 2P\bar{A}_e)\tilde{G}_{\alpha\beta\gamma\delta}$.

Again, as an immediate consequence of Corollary 3.3, we get following results:

Corollary 3.4. [17] *Let $M = \bar{M} \times_f \tilde{M}$ be a non-flat pseudosymmetric warped product manifold (i.e. $\nabla_l R_{hijk} = 2A_l R_{hijk} + A_h R_{lijk} + A_i R_{hljk} + A_j R_{hilk} + A_k R_{hijl}$). Then*

(1) *base is pseudosymmetric,*

(2) *fiber is*

  (i) *of constant curvature if $fA_e + f_e \neq 0$, and*

  (ii) *pseudosymmetric if*

$$P\big[2\tilde{A}_\epsilon \tilde{G}_{\alpha\beta\gamma\delta} + \tilde{A}_\alpha \tilde{G}_{\epsilon\beta\gamma\delta} + \tilde{A}_\beta \tilde{G}_{\alpha\epsilon\gamma\delta} + \tilde{A}_\gamma \tilde{G}_{\alpha\beta\epsilon\delta} + \tilde{A}_\delta \tilde{G}_{\alpha\beta\gamma\epsilon}\big] = 0.$$

Corollary 3.5. *If $M = \bar{M} \times_f \tilde{M}$ is a non-flat pseudosymmetric (in the sense of Chaki) warped product manifold with non-flat base of constant curvature, then pseudosymmetry and local symmetry are equivalent for M.*

Proof. Let M be non-flat pseudosymmetric manifold (in the sense of Chaki). Then

$$\nabla_l R_{hijk} = 2A_l R_{hijk} + A_h R_{lijk} + A_i R_{hljk} + A_j R_{hilk} + A_k R_{hijl}.$$

Now as base is non-flat and of constant curvature, Corollary 3.3 yields $\tilde{A}_\alpha = 0$ and

$$2\bar{A}_e\bar{R}_{abcd} + \bar{A}_a\bar{R}_{ebcd} + \bar{A}_b\bar{R}_{aecd} + \bar{A}_c\bar{R}_{abed} + \bar{A}_d\bar{R}_{abce} = 0, \tag{3.2}$$

which turns into

$$\bar{\nabla}_a\bar{R}_{abab} = 2\bar{A}_a\bar{R}_{abab} + \bar{A}_a\bar{R}_{abab} + \bar{A}_a\bar{R}_{abab}.$$

Since base is of non-flat, then from above we get $\bar{A}_a = 0$. Thus $A = 0$ on M, and M becomes locally symmetric. The converse part is obvious as every locally symmetric manifold is pseudosymmetric. Hence the theorem. □

Again, since a recurrent manifold is also a special case of weakly symmetric manifold, Theorem 3.1 leads to the following:

Corollary 3.6. *Let $M = \bar{M} \times_f \tilde{M}$ be a non-flat warped product manifold. Then M is recurrent (i.e. $\nabla_l R_{hijk} = A_l R_{hijk}$) if and only if the following conditions hold:*

(1) *base is recurrent (i.e. $\bar{\nabla}_e\bar{R}_{abcd} = \bar{A}_e\bar{R}_{abcd}$),*

(2) $\tilde{\nabla}_\epsilon\tilde{R}_{\alpha\beta\gamma\delta} - \tilde{A}_\epsilon\tilde{R}_{\alpha\beta\gamma\delta} = -fP\tilde{A}_\epsilon\tilde{G}_{\alpha\beta\gamma\delta}$ *and* $\bar{A}_e(\tilde{R}_{\alpha\beta\gamma\delta} - fP\tilde{G}_{\alpha\beta\gamma\delta}) = 0$,

(3) $\tilde{A}_\alpha\bar{R}_{abcd} = 0$,

(4) $\bar{\nabla}_e T_{ab} = \bar{A}_e T_{ab}$ *and* $T_{ab}\tilde{A}_\gamma = 0$,

(5) $f^d\bar{R}_{abcd} + (f_a)T_{bc} - (f_b)T_{ac} = 0$,

(6) $f_e\tilde{R}_{\alpha\beta\gamma\delta} = f^2P_e\tilde{G}_{\alpha\beta\gamma\delta}$.

4. Weakly Ricci symmetric warped product manifolds

Theorem 4.1. *Let $M = \bar{M} \times_f \tilde{M}$ be a non-Ricci-flat warped product manifold. Then M is WRS_n (i.e.*

$$\nabla_l S_{ij} = A_l S_{ij} + B_i S_{lj} + D_j S_{il}) \tag{4.1}$$

if and only if the following conditions hold:

(1) $\bar{\nabla}_e\bar{S}_{ab} - \bar{A}_e\bar{S}_{ab} - \bar{B}_a\bar{S}_{eb} - \bar{D}_b\bar{S}_{ae} = (n-p)(\nabla_e T_{ab} - \bar{A}_e T_{ab} - \bar{B}_a T_{eb} - \bar{D}_b T_{ae})$,

(2) $\tilde{\nabla}_\epsilon\tilde{S}_{\alpha\beta} - \tilde{A}_\epsilon\tilde{S}_{\alpha\beta} - \tilde{B}_\alpha\tilde{S}_{\epsilon\beta} - \tilde{D}_\beta\tilde{S}_{\alpha\epsilon} = Q(\tilde{A}_\epsilon\tilde{g}_{\alpha\beta} + \tilde{B}_\alpha\tilde{g}_{\epsilon\beta} + \tilde{D}_\beta\tilde{g}_{\alpha\epsilon})$,

(3) $\tilde{A}_\alpha\big[\bar{S}_{ab} - (n-p)T_{ab}\big] = \tilde{B}_\alpha\big[\bar{S}_{ab} - (n-p)T_{ab}\big] = \tilde{D}_\alpha\big[\bar{S}_{ab} - (n-p)T_{ab}\big] = 0$,

(4) $(f\bar{A}_e + f_e)\tilde{S}_{\alpha\beta} = (fQ_e - fQ\bar{A}_e - f_eQ)\tilde{g}_{\alpha\beta}$,
$(2f\bar{B}_e + f_e)\tilde{S}_{\alpha\beta} = \big[ff^c\big(S_{ce} - (n-p)T_{ce}\big) - Qf_e - 2fQ\bar{B}_e\big]\tilde{g}_{\alpha\beta}$,
$(2f\bar{D}_e + f_e)\tilde{S}_{\alpha\beta} = \big[ff^c\big(S_{ce} - (n-p)T_{ce}\big) - Qf_e - 2fQ\bar{D}_e\big]\tilde{g}_{\alpha\beta}$,

where

$$A_i = \begin{cases} \bar{A}_i & \text{for } \ i = 1, \dots, p \\ \tilde{A}_i & \text{otherwise,} \end{cases} \qquad B_i = \begin{cases} \bar{B}_i & \text{for } \ i = 1, \dots, p \\ \tilde{B}_i & \text{otherwise,} \end{cases}$$

$$D_i = \begin{cases} \bar{D}_i & \text{for } \ i = 1, \dots, p \\ \tilde{D}_i & \text{otherwise.} \end{cases}$$

Proof. Let M be a non-Ricci-flat weakly Ricci symmetric manifold. Then considering all possible cases of equation (4.1) for $h, i, j, k, l \in \big\{1, 2, \dots, p\big\} \cup \big\{p+1, p+2, \dots, n\big\}$ and by putting their values from (2.4) and (2.7), we get our assertion easily. □

As an immediate consequence of Theorem 4.1, we get the following results:

Corollary 4.1. *Let $M = \bar{M} \times_f \tilde{M}$ be a non-Ricci-flat weakly Ricci symmetric warped product manifold such that*

$$\nabla_l S_{ij} = A_l S_{ij} + B_i S_{lj} + D_j S_{il}.$$

Then

(1) *base is weakly Ricci symmetric if* $\nabla_e T_{ab} - A_e T_{ab} - B_a T_{eb} - D_b T_{ae} = 0$,

(2) *fiber is*

(i) *weakly Ricci symmetric if* $Q(\tilde{A}_\epsilon \tilde{g}_{\alpha\beta} + \tilde{B}_\alpha \tilde{g}_{\epsilon\beta} + \tilde{D}_\beta \tilde{g}_{\alpha\epsilon}) = 0$,

(ii) *Ricci symmetric if* $[\bar{S}_{ab} - (n-p)T_{ab}] \neq 0$, *and*

(iii) *Einstein if any one of*
(a) $2f\bar{A}_e + f_e$ (b) $2f\bar{B}_e + f_e$ (c) $2f\bar{A}_e + f_e$ *is non-zero.*

Corollary 4.2. *Let $M = \bar{M} \times \tilde{M}$ be a not locally symmetric decomposable manifold. Then M is WRS_n if and only if one of the decomposition is weakly Ricci symmetric and the other is Ricci-flat.*

Proof. A warped product manifold is decomposable if the warping function $f \equiv 1$. Then the conditions of Theorem 4.1 reduce to

$$\bar{\nabla}_e \bar{S}_{ab} - \bar{A}_e \bar{S}_{ab} - \bar{B}_a \bar{S}_{eb} - \bar{D}_b \bar{S}_{ae} = 0,$$

$$\tilde{\nabla}_\epsilon \tilde{S}_{\alpha\beta} - \tilde{A}_\epsilon \tilde{S}_{\alpha\beta} - \tilde{B}_\alpha \tilde{S}_{\epsilon\beta} - \tilde{D}_\beta \tilde{S}_{\alpha\epsilon} = 0,$$

$$\tilde{A}_\alpha \bar{S}_{ab} = 0 \text{ and } \bar{A}_e \tilde{S}_{\alpha\beta} = 0.$$

Now for a decomposable manifold, we have $S_{ab} = \bar{S}_{ab}$, $S_{\alpha\beta} = \tilde{S}_{\alpha\beta}$. Then from the above reduced conditions the result follows. □

Since every pseudo Ricci symmetric manifold in the sense of Chaki [7] is a WRS_n, so Theorem 4.1 leads to the following:

Corollary 4.3. *Let $M = \bar{M} \times_f \tilde{M}$ be a non-Ricci-flat warped product. Then M is pseudo Ricci symmetric in the sense of Chaki (i.e. $\nabla_l S_{ij} = 2A_l S_{ij} + A_i S_{lj} + A_j S_{il}$) if and only if the following conditions hold:*

(1) $\bar{\nabla}_e \bar{S}_{ab} - 2\bar{A}_e \bar{S}_{ab} - \bar{A}_a \bar{S}_{eb} - \bar{A}_b \bar{S}_{ae} = (n-p)(\nabla_e T_{ab} - 2A_e T_{ab} - A_a T_{eb} - A_b T_{ae})$,

(2) $\tilde{\nabla}_\epsilon \tilde{S}_{\alpha\beta} - 2\tilde{A}_\epsilon \tilde{S}_{\alpha\beta} - \tilde{A}_\alpha \tilde{S}_{\epsilon\beta} - \tilde{A}_\beta \tilde{S}_{\alpha\epsilon} = Q(2\tilde{A}_\epsilon \tilde{g}_{\alpha\beta} + \tilde{A}_\alpha \tilde{g}_{\epsilon\beta} + \tilde{A}_\beta \tilde{g}_{\alpha\epsilon})$,

(3) $\tilde{A}_\alpha(\bar{S}_{ab} - (n-p)T_{ab}) = 0$,

(4) $(2f\bar{A}_e + f_e)\tilde{S}_{\alpha\beta} = (fQ_e - 2fQ\bar{A}_e - f_e Q)\tilde{g}_{\alpha\beta} = (ff^c(S_{ce} - (n-p)T_{ce}) - Qf_e - 2fQA_e)\tilde{g}_{\alpha\beta}$.

Again as an immediate consequence of Corollary 4.3, we get the following result:

Corollary 4.4 ([17])**.** *Let $M = \bar{M} \times_f \tilde{M}$ be a non-flat pseudo Ricci symmetric warped product manifold (i.e. $\nabla_l S_{ij} = 2A_l S_{ij} + A_i S_{lj} + A_j S_{il}$). Then*

(1) *base is pseudo Ricci symmetric if $\nabla_e T_{ab} - A_e T_{ab} - A_a T_{eb} - A_b T_{ae} = 0$,*

(2) *fiber is*

  (i) *pseudo Ricci symmetric if $Q(2\tilde{A}_\epsilon \tilde{g}_{\alpha\beta} + \tilde{A}_\alpha \tilde{g}_{\epsilon\beta} + \tilde{A}_\beta \tilde{g}_{\alpha\epsilon}) = 0$,*

  (ii) *Ricci symmetric if $(\bar{S}_{ab} - (n-p)T_{ab}) \neq 0$, and*

  (iii) *Einstein if $(2f\bar{A}_e + f_e) \neq 0$.*

Since the class of Ricci recurrent manifolds is a subclass of weakly Ricci symmetric manifolds, Theorem 4.1 leads to the following:

Corollary 4.5. *Let $M = \bar{M} \times_f \tilde{M}$ be a non-Ricci-flat warped product manifold. Then M is a Ricci recurrent manifold ($\nabla_l S_{ij} = A_l R_{ij}$) if and only if the following conditions hold:*

(1) $\bar{\nabla}_e \bar{S}_{ab} - \bar{A}_e \bar{S}_{ab} = (n-p)(\nabla_e T_{ab} - A_e T_{ab})$,

(2) $\tilde{\nabla}_\epsilon \tilde{S}_{\alpha\beta} - \tilde{A}_\epsilon \tilde{S}_{\alpha\beta} = Q\tilde{A}_\epsilon \tilde{g}_{\alpha\beta}$,

(3) $\tilde{A}_\alpha[\bar{S}_{ab} - (n-p)T_{ab}] = 0$,

(4) $(f\bar{A}_e + f_e)\tilde{S}_{\alpha\beta} = (fQ_e - fQ\bar{A}_e - f_e Q)\tilde{g}_{\alpha\beta}$,
$f_a \tilde{S}_{\alpha\beta} = [ff^c(\tilde{S}_{ac} - (n-p)T_{ac}) - Qf_a]\tilde{g}_{\alpha\beta}$.

5. Examples of WS_n and WRS_n warped product manifolds

Example 1. We consider the warped product manifold $M = \bar{M} \times_f \tilde{M}$, where $(\bar{M}, \bar{g})$ is given by Example 2, $f(x^1, x^2, x^3) = e^{x^1+x^3}$ is a smooth function on $\bar{M}$ and $(\tilde{M}, \tilde{g})$ be a connected semi-Riemannian manifold of dimension 4, endowed with the metric

$$\tilde{g}_{44} = -1, \quad \tilde{g}_{55} = -x^5 e^{x^4}, \quad \tilde{g}_{66} = -x^6 e^{x^4}, \quad \tilde{g}_{77} = -x^7 e^{x^4},$$
$$\tilde{g}_{ij} = 0 \quad \text{for} \quad i \neq j, \; i, j = 4, 5, 6, 7.$$

Then the non-zero components of curvature tensor $\tilde{R}$ (upto symmetry) and Ricci tensor $\tilde{S}$ of $\tilde{M}$ are given by

$$\tilde{R}_{4545} = -\frac{1}{4} x^5 e^{x^4}, \quad \tilde{R}_{4646} = -\frac{1}{4} x^6 e^{x^4}, \quad \tilde{R}_{4747} = -\frac{1}{4} x^7 e^{x^4},$$
$$\tilde{R}_{5656} = -\frac{1}{4} x^5 x^6 e^{2x^4}, \quad \tilde{R}_{5757} = -\frac{1}{4} x^5 x^7 e^{2x^4}, \quad \tilde{R}_{6767} = -\frac{1}{4} x^6 x^7 e^{2x^4},$$
$$\text{and} \quad \tilde{S}_{44} = -\frac{3}{4}, \quad \tilde{S}_{55} = -\frac{3}{4} x^5 e^{x^4}, \quad \tilde{S}_{66} = -\frac{3}{4} x^6 e^{x^4}, \quad \tilde{S}_{77} = -\frac{3}{4} x^7 e^{x^4}.$$

Then $\tilde{M}$ is a manifold of constant curvature with scalar curvature $\tilde{\kappa} = 3$. The warped product manifold M is of dimension 7 endowed with the metric g, whose non-zero components are given by

$$g_{11} = e^{x^1}, \quad g_{22} = e^{x^1}, \quad g_{33} = e^{x^1+x^3}, \quad g_{44} = -e^{x^1+x^3},$$
$$g_{55} = -x^5 e^{x^1+x^3+x^4}, \quad g_{66} = -x^6 e^{x^1+x^3+x^4}, \quad g_{77} = -x^7 e^{x^1+x^3+x^4}.$$

Then the non-zero components of curvature tensor (upto symmetry) of (M, g) are given by

$$R_{2i2i} = \frac{1}{4} g_{ii} \quad \text{for } i = 3, 4, \ldots, 7, \qquad R_{3i3i} = \frac{1}{4} g_{ii} e^{x^3} \quad \text{for } i = 4, 5, \ldots, 9,$$
$$R_{4i4i} = -\frac{1}{4} g_{ii} \quad \text{for } i = 5, 6, 7,$$
$$R_{5i5i} = -\frac{1}{4} x^5 g_{ii} e^{x^3+x^4} \quad \text{for } i = 6, 7, \qquad R_{6i6i} = -\frac{1}{4} x^6 g_{ii} e^{x^3+x^4} \quad \text{for } i = 7.$$

Then the non-zero components of covariant derivatives of R (upto symmetry) are given by

$$R_{1jji,i} = -\frac{1}{2}R_{jiji} \quad \text{and} \quad R_{2i2i,1} = R_{2i2i} \quad \text{for } i,j = 2,3,\dots,7.$$

The only non-zero components of Ricci tensor and its covariant derivatives (upto symmetry) are given by

$$S_{ii} = -\frac{5}{4}e^{-x^1}g_{ii} \quad \text{for } i = 2,3,\dots,7.$$

$$S_{1i,i} = \frac{1}{2}S_{ii} \quad \text{and} \quad S_{ii,1} = S_{ii} \quad \text{for all } i.$$

Then it is clear that the manifold M satisfies the defining condition (3.1) for all $h,i,j,k,l \in \{1,2,\dots,9\}$ with

$$A_i = \begin{cases} -1 & \text{for } \ i=1 \\ 0 & \text{otherwise,} \end{cases} \qquad B_i = \begin{cases} -\frac{1}{2} & \text{for } \ i=1 \\ 0 & \text{otherwise,} \end{cases}$$

$$D_i = \begin{cases} -\frac{1}{2} & \text{for } \ i=1 \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Thus the warped product manifold $M = \bar{M} \times_f \tilde{M}$ is WS_7.

Example 2. Let $(\bar{M}, \bar{g})$ be a 3-dimensional connected semi-Riemannian manifold endowed with the metric

$$\bar{g}_{11} = \bar{g}_{22} = e^{x^1}, \quad \bar{g}_{33} = e^{x^1+x^3}, \quad \bar{g}_{ij} = 0 \quad \text{for } i \neq j, \ i,j = 1,2,3$$

and $f(x^1,x^2,x^3) = \frac{2}{5}e^{x^1+x^3}$ be a smooth function on $\bar{M}$. Then the only non-zero components of curvature tensor $\bar{R}$ (upto symmetry) and its covariant derivatives are given by

$$\bar{R}_{2323} = \frac{e^{x^1+x^3}}{4}$$

$$\bar{R}_{1223,3} = \frac{e^{x^1+x^3}}{8}, \quad \bar{R}_{1323,2} = -\frac{e^{x^1+x^3}}{8}, \quad \bar{R}_{2323,1} = -\frac{e^{x^1+x^3}}{4}.$$

The only non-zero components of Ricci tensor, its covariant derivatives (upto symmetry) and scalar curvature are given by

$$\bar{S}_{22} = -\frac{1}{4}, \quad \bar{S}_{33} = -\frac{e^{x^3}}{4},$$
$$\bar{S}_{12,2} = \frac{1}{8}, \quad \bar{S}_{13,3} = \frac{e^{x^3}}{8}, \quad \bar{S}_{22,1} = \frac{1}{4}, \quad \bar{S}_{33,1} = \frac{e^{x^3}}{4}$$
$$\text{and} \quad \bar{\kappa} = -\frac{e^{-x^1}}{2}.$$

Again let $(\tilde{M}, \tilde{g})$ be a 6-dimensional connected semi-Riemannian manifold endowed with the metric

$$\tilde{g}_{44} = -1, \quad \tilde{g}_{55} = -e^{x^4}, \quad \tilde{g}_{66} = -e^{x^4}(x^5)^2,$$
$$\tilde{g}_{77} = -1, \quad \tilde{g}_{88} = -e^{x^7}, \quad \tilde{g}_{99} = -e^{x^7}(x^8)^2, \quad \tilde{g}_{ij} = 0 \quad \text{for } i \neq j, \ i, j = 4, 5, \dots, 9.$$

Then the non-zero components of curvature tensor $\tilde{R}$ (upto symmetry) are

$$\tilde{R}_{4545} = -\frac{e^{x^4}}{4}, \quad \tilde{R}_{4646} = -\frac{e^{x^4}(x^5)^2}{4}, \quad \tilde{R}_{5656} = -\frac{e^{2x^4}(x^5)^2}{4},$$
$$\tilde{R}_{7878} = -\frac{e^{x^7}}{4}, \quad \tilde{R}_{7979} = -\frac{e^{x^7}(x^8)^2}{4}, \quad \tilde{R}_{8989} = -\frac{e^{2x^7}(x^8)^2}{4}$$

and the non-zero components of Ricci tensor and scalar curvature are

$$\tilde{S}_{44} = -\frac{1}{2}, \quad \tilde{S}_{55} = -\frac{e^{x^4}}{2}, \quad \tilde{S}_{66} = -\frac{e^{x^4}(x^5)^2}{2}, \quad \tilde{S}_{77} = -\frac{1}{2}, \quad \tilde{S}_{88} = -\frac{e^{x^7}}{2},$$
$$\tilde{S}_{99} = -\frac{e^{x^7}(x^8)^2}{2}, \quad \tilde{\kappa} = 3.$$

Then this manifold $\tilde{M}$ is locally symmetric and Einstein but not of constant curvature. Now the warped product manifold $M = \bar{M} \times_f \tilde{M}$ is of dimension 9 endowed with the metric $g = \bar{g} \times_f \tilde{g}$, whose non-zero components are given by

$$g_{11} = e^{x^1}, \quad g_{22} = e^{x^1}, \quad g_{33} = e^{x^1+x^3}, \quad g_{44} = -\frac{2}{5}e^{x^1+x^3}, \quad g_{55} = -\frac{2}{5}e^{x^1+x^3+x^4}$$
$$g_{66} = -\frac{2}{5}e^{x^1+x^3+x^4}(x^5)^2, \quad g_{77} = -\frac{2}{5}e^{x^1+x^3}, \quad g_{88} = -\frac{2}{5}e^{x^1+x^3+x^7},$$

$$g_{99} = -\frac{2}{5}e^{x^1+x^3+x^7}(x^8)^2.$$

Again, the only non-zero components of curvature tensor R (upto symmetry) are given by

$$R_{2i2i} = \frac{1}{4}g_{ii} \quad \text{for } i = 3, 4, \ldots, 9, \qquad R_{3i3i} = \frac{1}{4}g_{ii}e^{x^3} \quad \text{for } i = 4, 5, \ldots, 9,$$

$$R_{4i4i} = \begin{cases} -\dfrac{1}{20}g_{ii}(2e^{x^3} - 3) & \text{for } \ i = 5, 6, \\ -\dfrac{1}{10}g_{ii}(e^{x^3} + 1) & \text{for } i = 7, 8, 9, \end{cases}$$

$$R_{5i5i} = \begin{cases} -\dfrac{1}{20}g_{ii}e^{x^4}(2e^{x^3} - 3) & \text{for } i = 6, \\ -\dfrac{1}{10}g_{ii}e^{x^4}(e^{x^3} + 1) & \text{for } i = 7, 8, 9, \end{cases}$$

$$R_{6i6i} = -\frac{1}{10}g_{ii}e^{x^4}(e^{x^3} + 1)(x^5)^2 \quad \text{for } i = 7, 8, 9,$$

$$R_{7i7i} = -\frac{1}{20}g_{ii}(2e^{x^3} - 3) \quad \text{for } i = 8, 9, \qquad R_{8989} = -\frac{1}{20}g_{ii}e^{x^7}(2e^{x^3} - 3).$$

The non-zero components of covariant derivatives of R (upto symmetry) are given by

$$R_{1jji,i} = -\frac{1}{2}R_{jiji} \quad \text{for } i, j = 2, 3, \ldots, 9 \text{ and } i \neq j,$$

$$R_{jiji,1} = R_{jiji} \quad \text{for } i, j = 2, 3, \ldots, 9 \text{ and } i \neq j,$$

$$R_{344i,i} = \begin{cases} \dfrac{3}{40}g_{ii} & \text{for } i = 5, 6, \\ -\dfrac{1}{20}g_{ii} & \text{for } i = 7, 8, 9, \end{cases} \qquad R_{355i,i} = \begin{cases} \dfrac{3}{40}e^{x^4}g_{ii} & \text{for } i = 4, 6, \\ -\dfrac{1}{20}e^{x^4}g_{ii} & \text{for } i = 7, 8, 9, \end{cases}$$

$$R_{366i,i} = \begin{cases} \dfrac{3}{40}e^{x^4}g_{ii}(x^5)^2 & \text{for } i = 4, 5, \\ -\dfrac{1}{20}e^{x^4}g_{ii}(x^5)^2 & \text{for } i = 7, 8, 9, \end{cases} \qquad R_{377i,i} = \begin{cases} -\dfrac{1}{20}g_{ii} & \text{for } i = 4, 5, 6, \\ \dfrac{3}{40}g_{ii} & \text{for } i = 8, 9, \end{cases}$$

$$R_{388i,i} = \begin{cases} -\dfrac{1}{20}e^{x^7}g_{ii} & \text{for } i = 4, 5, 6, \\ \dfrac{3}{40}e^{x^7}g_{ii} & \text{for } i = 7, 9, \end{cases}$$

$$R_{399i,i} = \begin{cases} -\dfrac{1}{20}e^{x^7}g_{ii}(x^8)^2 & \text{for } i = 4,5,6, \\ \dfrac{3}{40}e^{x^7}g_{ii}(x^8)^2 & \text{for } i = 7,8, \end{cases}$$

$$R_{4i4i,3} = \begin{cases} -\dfrac{3}{20}g_{ii} & \text{for } i = 5,6, \\ \dfrac{1}{10}g_{ii} & \text{for } i = 7,8,9, \end{cases} \qquad R_{5i5i,3} = \begin{cases} -\dfrac{3}{20}e^{x^4}g_{ii} & \text{for } i = 6, \\ \dfrac{1}{10}e^{x^4}g_{ii} & \text{for } i = 7,8,9, \end{cases}$$

$$R_{6i6i,3} = \frac{1}{10}e^{x^4}g_{ii}(x^5)^2 \quad \text{for } i = 7,8,9,$$

$$R_{7i7i,3} = -\frac{3}{40}g_{ii} \quad \text{for } i = 8,9, \qquad R_{8i8i,3} = -\frac{3}{40}e^{x^7}g_{ii} \quad \text{for } i = 9.$$

The only non-zero components of Ricci tensor and its covariant derivatives (upto symmetry) are given by

$$S_{ii} = -\frac{7}{4}e^{-x^1}g_{ii} \quad \text{for } i = 2,3,\ldots,9,$$
$$S_{1i,i} = \frac{1}{2}S_{ii} \quad \text{and} \quad S_{ii,1} = S_{ii} \quad \text{for all } i.$$

Then it is clear that in the manifold M satisfies the defining condition (4.1) for all $l,i,j \in \{1,2,\ldots,9\}$ with

$$A_i = \begin{cases} -1 & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases} \qquad B_i = \begin{cases} -\dfrac{1}{2} & \text{for } i = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$D_i = \begin{cases} -\dfrac{1}{2} & \text{for } i = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

Thus the warped product manifold $M = \bar{M} \times_f \tilde{M}$ is WRS_9.

Remark. We note that the warped product M given in Example 1 is also WRS_7, and the warped product given in Example 2 is WRS_9 but not WS_9.

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