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The Zermelo conditions and higher order homogeneous functions

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Abstract. Invariance under reparametrizations of integral curves of higher order differential equations, including variational equations related to Finsler geometry, is studied. The classical homogeneity concepts are introduced within the theory of (jet) differential groups, known in the theory of differential invariants. On this basis the well-known generalizations of the Euler theorem are obtained (the Zermelo conditions). It is shown that every integral curve of a system of differential equations whose left-hand sides are higher order positive homogeneous functions, is invariant with respect to all reparametrizations, i.e. a set solution. Then the positive homogeneity concept is applied to second order variational equations. We show that the systems with positive homogeneous Lagrangians are defined by positive homogeneous functions, and vice versa.

1. Introduction

In this work we consider the problem of possible higher order generalizations of the Finsler geometry, based on higher order homogeneous Finsler fundamental functions (Lagrangians). An alternative setting and understanding of this

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problem includes the definitions and analysis of the properties of functions, invariant with respect to diffeomorphisms of their domains of definition, and their independence of parametrization.

It should be pointed out, that the geodesic equations in the Riemann and Finsler geometry are *variational* and they are derived from the first order Lagrangians, depending on curves and their tangent vectors; the Lagrangians satisfy the classical *positive homogeneity condition* for functions.

For basic Finsler geometry, related to the topics of this paper, we refer to McKiernan [14], Crampin and Saunders [2], and Kawaguchi [7].

In this paper we analyse the relation of the homogeneity concept and the invariance of functions with respect to the parameter changes. We use to this purpose an elementary version of the Ehresmann's theory of jets, differential groups and jet prolongations of curves and tangent spaces. For sources, well adapted to this *jet approach*, we refer to Ehresmann [3], [4], Grigore and Krupka [6], D. Krupka and M. Krupka [9], and Krupka and Urban [10].

Our main result consists in introducing, on the basis of the theory of jets, the positive homogeneity concept for functions, depending on curves and their derivatives of an arbitrary finite order. It turns out, in particular, that the standard *positive homogeneity condition* for a function $f = f(x^K, \dot{x}^K)$, depending on curves $t \to x^K(t)$ and their derivatives $t \to \dot{x}^K(t)$, expressed by the *Euler formula*

$$\frac{\partial f}{\partial \dot{x}^K}\dot{x}^K = f, \tag{1}$$

should be replaced by not only one condition, a higher order analogue of (1), but also additional conditions appear.

If for example, $f = f(x^K, \dot{x}^K, \ddot{x}^K)$, where $\dot{x}^K$ and $\ddot{x}^K$ are the first and the second derivative variables, then the generalized positive homogeneous conditions read

$$\frac{\partial f}{\partial \dot{x}^K}\dot{x}^K + 2\frac{\partial f}{\partial \ddot{x}^K}\ddot{x}^K = f, \quad \frac{\partial f}{\partial \ddot{x}^K}\dot{x}^K = 0 \tag{2}$$

(summation through double indices).

The higher order analogues of conditions (1), (2), originally described by Zermelo [19] as the necessary conditions for the variational integral to be invariant under reparametrizations, are known as the *Zermelo conditions*, see e.g. McKiernan [14], Kawaguchi [7], Kondo [8], Matsyuk [13], Miron [15], P. Popescu and M. Popescu [17]. The problem of parameter invariance was considered in more general situation by McKiernan [14] who completed sufficiency of the Zermelo conditions in field theory; our results agree with his formulas. However, our definition of higher order positive homogeneity differs from that one due to Miron [15], and Miron, Anastasiei and Bucataru [16] (the parameter-invariance formulas are different). Parameter-invariance problem was also discussed by Matsyuk [13] in terms of generating vector fields. An introduction to the parameter invariant variational problems can be found in e.g. Logan [12].

In this paper we prove, by the methods of *jet groups*, a complete theorem, characterizing the Zermelo conditions as higher order homogeneous functions. We call the corresponding result the Euler–Zermelo theorem.

The higher order positive homogeneity concept is then applied to systems of higher order differential equations; the systems *need not* be variational. We show that all solutions of a system of differential equations, given by positively homogeneous functions, are parameter-independent.

Finally, we consider second order variational equations, defined on velocity manifolds. It is shown that systems with a positive homogeneous Lagrangian are defined by positive homogeneous functions, and conversely, systems defined by positive homogeneous functions admit a positive homogeneous Lagrangian.

The ideas and the proofs we propose can be extended to functions of more independent variables, and to differential equations on manifolds. In particular, one can state in this way the jet foundations of the geometry of general Kawaguchi spaces.

2. Regular velocities

In this section we associate to curves in a smooth manifold a space on which reparametrizations of curves act as a (finite-dimensional) Lie group (the *differential group*). Invariants of this Lie group action correspond with parameter-invariant functions or parameter-equivariant mappings. Basic theoretical constructions we use are special cases of general higher order theory of contact elements of mappings of the Euclidean spaces $\mathbb{R}^n$ into m-dimensional manifolds (see e.g. [3], [4], [6], [9], [10]).

Throughout this section, we consider curves in the Euclidean space of dimension $m+1$, $\mathbb{R}^{m+1}$, where $m \geq 1$. The *canonical coordinates* on $\mathbb{R}^{m+1}$ are denoted by y^K, where $K = 1, 2, \dots, m+1$.

By a *velocity of order* r at a point $y \in \mathbb{R}^{m+1}$ we mean an r-jet $P = J^r_0\zeta$ with source $0 \in \mathbb{R}$ and target $\zeta(0) = y$. Given a representative ζ of the equivalence class P, we can identify P in the well-known sense with the ordered collection of numbers $(y^K(P), y^K_1(P), \dots, y^K_r(P))$, defined by the derivatives of the curve

$t \to y^K \zeta(t)$ at the origin $0 \in \mathbb{R}$,

$$y_l^K(P) = D^l(y^K \zeta)(0). \tag{3}$$

The set of velocities at $y \in \mathbb{R}^{m+1}$ is denoted $J^r_{(0,y)}(\mathbb{R}, \mathbb{R}^{m+1})$. Velocities of order r are also called *tangent vectors* of order r. We denote

$$T^r\mathbb{R}^{m+1} = \bigcup_{y \in \mathbb{R}^{m+1}} J^r_{(0,y)}(\mathbb{R}, \mathbb{R}^{m+1}),$$

and define surjective mappings $\tau^{r,s} : T^r\mathbb{R}^{m+1} \to T^s\mathbb{R}^{m+1}$, where $0 \leq s \leq r$, by $\tau^{r,s}(J^r_0 \zeta) = J^s_0 \zeta$.

We consider the set of velocities of order r $T^r\mathbb{R}^{m+1}$ with standard geometric structures. The functions $(y^K, y_1^K, y_2^K, \ldots, y_r^K)$, defined by formula (3), are the *canonical coordinates* on $T^r\mathbb{R}^{m+1}$. Sometimes we also use in computations an equivalent formula $y_l^K(P) = D^l(\mathrm{tr}^K_{\psi\zeta(0)} \psi\zeta)(0)$, where $\psi = (y^K)$ and tr^K_ξ is the K-component of the translation

$$\mathbb{R}^{m+1} \ni x \to \mathrm{tr}_\xi(x) = x - \xi \in \mathbb{R}^{m+1}$$

of $\mathbb{R}^{m+1}$. The *canonical trivialization* of $T^r\mathbb{R}^{m+1}$ is the mapping

$$T^r\mathbb{R}^{m+1} \in J^r_0 \zeta \to (\zeta(0), J^r_0 \, \mathrm{tr}_{\psi\zeta(0)} \, \psi\zeta) \in \mathbb{R}^{m+1} \times J^r_{(0,0)}(\mathbb{R}, \mathbb{R}^{m+1}). \tag{4}$$

In particular, the mapping (4) shows that $T^r\mathbb{R}^{m+1}$ is a trivial vector bundle with base $\mathbb{R}^{m+1}$, projection $\tau^{r,0} : T^r\mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$, and type fibre $J^r_{(0,0)}(\mathbb{R}, \mathbb{R}^{m+1})$. We call $T^r\mathbb{R}^{m+1}$ the *bundle of velocities of order* r over $\mathbb{R}^{m+1}$.

We now recall the definition of the r-th differential group of the real line $\mathbb{R}$. Let r be a positive integer, and consider the manifold of r-jets with source and target $0 \in \mathbb{R}$, $J^r_{(0,0)}(\mathbb{R}, \mathbb{R})$. We set for every r-jet $A \in J^r_{(0,0)}(\mathbb{R}, \mathbb{R})$, $A = J^r_0 \alpha$,

$$a_l(A) = D^l \alpha(0). \tag{5}$$

Formula (5) defines real functions $a_l : J^r_{(0,0)}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$, where $l = 1, 2, \ldots, r$. These functions constitute a coordinate system, and are called the *canonical coordinates* on $J^r_{(0,0)}(\mathbb{R}, \mathbb{R})$.

The set $J^r_{(0,0)}(\mathbb{R}, \mathbb{R})$ contains the subset of *invertible* r-jets, the r-jets of *immersions*

$$L^r = \mathrm{Imm}\, J^r_{(0,0)}(\mathbb{R}, \mathbb{R}) = \{J^r_0 \alpha \in J^r_{(0,0)}(\mathbb{R}, \mathbb{R}) \mid a_1(J^r_0 \alpha) \neq 0\}.$$

L^r is dense and open subset in $J^r_{(0,0)}(\mathbb{R}, \mathbb{R})$. Restricting the functions $a_l : J^r_{(0,0)}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ we get the *canonical coordinates* on L^r.

The composition of jets defines a mapping

$$L^r \times L^r \ni (A, B) \to A \circ B \in L^r, \tag{6}$$

which is easily seen to be a group multiplication. Explicitly, if $A = J_0^r \alpha$ and $B = J_0^r \beta$, then $A \circ B = J_0^r(\alpha \circ \beta)$. The set L^r with this group multiplication has a Lie group structure, and is called the *r-th differential group* of $\mathbb{R}$.

Let a positive integer l be given, and let the symbol

$$\sum_{(I_1, I_2, \dots, I_p)}$$

denote the summation through all partitions $(I_1, I_2, \dots, I_p)$ of the set $\{1, 1, \dots, 1\}$ (l elements). By $|I_k|$ we mean the number of elements of I_k.

Using the canonical coordinates we can easily describe the multiplication (6) explicitly. Denote $C = A \circ B = J_0^r(\alpha \circ \beta)$, and write in the canonical coordinates $a_k = a_k(A)$, $b_k = b_k(B)$, and $c_k = c_k(C)$.

Lemma 1. *The group multiplication $(A, B) \to A \circ B$ has the equations*

$$c_l = \sum_{p=1}^{l} a_p \sum_{(I_1, I_2, \dots, I_p)} b_{|I_1|} b_{|I_2|} \dots b_{|I_p|}. \tag{7}$$

Proof. To obtain formula (7), we use definition (6) of the r-jet $J_0^r(\alpha \circ \beta)$, and apply the chain rule and the formula for differentiating of a product. $\square$

The differential group L^r acts on $T^r \mathbb{R}^{m+1}$ to the right by composition of jets

$$(J_0^r \zeta, J_0^r \alpha) \to J_0^r \zeta \circ J_0^r \alpha = J_0^r(\zeta \circ \alpha);$$

sometimes we express this action as

$$T^r \mathbb{R}^{m+1} \times L^r \ni (P, A) \to P \circ A = J_0^r(\zeta \circ \alpha) \in T^r \mathbb{R}^{m+1}, \tag{8}$$

where $P = J_0^r \zeta$ and $A = J_0^r \alpha$. In order to describe this action in canonical coordinates, we denote

$$\bar{y}_l^K(J_0^r \zeta) = y_l^K(J_0^r \zeta \circ J_0^r \alpha).$$

Lemma 2. *The group action $(J_0^r \zeta, J_0^r \alpha) \to J_0^r \zeta \circ J_0^r \alpha$ is expressed by the equations*

$$\bar{y}^K = y^K, \quad \bar{y}_l^K = \sum_{p=1}^{l} y_p^K \sum_{(I_1, I_2, \dots, I_p)} a_{|I_1|} a_{|I_2|} \dots a_{|I_p|}. \tag{9}$$

Proof. Formula (9) follows from the definition (8), the chain rule and from the formula for differentiating of a product. □

We introduce the concept of the *prolongation* of a curve in $\mathbb{R}^{m+1}$ to a curve in the manifold of velocities $T^r\mathbb{R}^{m+1}$.

Let γ be a differentiable curve in $\mathbb{R}^{m+1}$, defined on an open interval $I \subset \mathbb{R}$. Any point $t \in I$ gives rise to a curve $s \to \gamma \circ \mathrm{tr}_{-t}(s)$, defined on a neighbourhood of the origin 0, and to the r-jet $J^r_0(\gamma \circ \mathrm{tr}_{-t})$; we get the curve

$$I \in t \to T^r\gamma(t) = J^r_0(\gamma \circ \mathrm{tr}_{-t}) \in T^r\mathbb{R}^{m+1}, \tag{10}$$

called the *r-jet prolongation* of the curve γ.

Note that for every isomorphism $\mu : J \to I$ of open intervals, and every point $s \in J$, the r-jet $J^r_0(\mathrm{tr}_{\mu(s)} \circ \mu \circ \mathrm{tr}_{-s})$ belongs to the differential group L^r; denote

$$\mu_s = \mathrm{tr}_{\mu(s)} \circ \mu \circ \mathrm{tr}_{-s}, \quad \mu^r(s) = J^r_0\mu_s. \tag{11}$$

The following theorem characterizes basic properties of the r-jet prolongation of a curve; in particular, we get a formula for reparametrizations of the domain of definition of the prolonged curve.

Theorem 1. *The r-jet prolongation $T^r\gamma$ of a curve $\gamma : I \to \mathbb{R}^{m+1}$ has the following properties:*

(a) *The expression of $T^r\gamma$ in canonical coordinates satisfies the recurrence formula*
$$y^K_l \circ T^r\gamma(t) = D(y^K_{l-1} \circ T^r\gamma)(t). \tag{12}$$

(b) *For any diffeomorphism $\mu : J \to I$ the mapping $T^r\gamma$ satisfies*
$$T^r(\gamma \circ \mu)(s) = T^r\gamma(\mu(s)) \circ \mu^r(s). \tag{13}$$

Proof. (a) Clearly, we have $y^K_l \circ T^r\gamma(t) = D^l(y^K\gamma)(t) = D(D^{l-1}(y^K\gamma))(t)$, proving (12).

(b) Since by (10), $T^r(\gamma \circ \mu)(s) = J^r_0(\gamma \circ \mathrm{tr}_{-\mu(s)} \circ \mathrm{tr}_{\mu(s)} \circ \mu \circ \mathrm{tr}_{-s})$, we have

$$T^r(\gamma \circ \mu)(s) = T^r\gamma(\mu(s)) \circ J^r_0(\mathrm{tr}_{\mu(s)} \circ \mu \circ \mathrm{tr}_{-s}),$$

proving (13). □

Formula (13) shows that the curve $s \to T^r(\gamma \circ \mu)(s)$ in uniquely determined by the curve $s \to T^r\gamma(s)$ and a family of group elements $\mu^r(s) \in L^r$, derived from μ.

We now restrict our attention to curves that are *immersions*, i.e., the curves whose tangent vectors are never zero. A velocity $P \in T^r\mathbb{R}^{m+1}$ is said to be *regular*, if $P = J_0^r\zeta$ for an immersion ζ. Regular velocities form an open, dense, and L^r-invariant set in $T^r\mathbb{R}^{m+1}$, denoted by $\operatorname{Imm} T^r\mathbb{R}^{m+1}$.

Restricting the canonical coordinates $(y^K, y_1^K, y_2^K, \dots, y_r^K)$, we get the *canonical coordinates* on the set $\operatorname{Imm} T^r\mathbb{R}^{m+1}$. But from the definition of an immersion it follows that for every point $P \in \operatorname{Imm} T^r\mathbb{R}^{m+1}$, at least one of the coordinates $y_1^1(P), y_1^2(P), \dots, y_1^{m+1}(P)$ is different from 0. We set for every element L of the sequence $(1, 2, \dots, m+1)$

$$V^{r,L} = \{P \in T^r\mathbb{R}^{m+1} \mid y_1^L(P) \neq 0\}.$$

$V^{r,L}$ is an open set in $\operatorname{Imm} T^r\mathbb{R}^{m+1}$ and in $T^r\mathbb{R}^{m+1}$, endowed with the *canonical coordinates* $(y^K, y_1^K, y_2^K, \dots, y_r^K)$, satisfying $y_1^L \neq 0$.

Now consider the group action $(P, A) \to P \circ A$ of L^r on $\operatorname{Imm} T^r\mathbb{R}^{m+1}$, and the equivalence relation $\mathfrak{R}$ on $\operatorname{Imm} T^r\mathbb{R}^{m+1}$ "*there exists* A *such that* $Q = P \circ A$".

Lemma 3. *Let* (P, Q) *be a point of the set* $\operatorname{Imm} T^r\mathbb{R}^{m+1} \times \operatorname{Imm} T^r\mathbb{R}^{m+1}$. *The following two conditions are equivalent:*

(a) $(P, Q) \in \mathfrak{R}$.

(b) *There exists an index* L, $1 \leq L \leq m+1$, *and an element* $A \in L^r$ *such that* $(P, Q) \in V^{r,L}$, *and the coordinates* $y_l^K = y_l^K(P)$, $\bar{y}_l^K = \bar{y}_l^K(Q)$, $a_l = a_l(A)$ *satisfy*

$$\bar{y}^K = y^K, \quad \bar{y}_l^\sigma = \sum_{p=1}^{l} y_p^\sigma \sum_{(I_1, I_2, \dots, I_p)} a_{|I_1|} a_{|I_2|} \dots a_{|I_p|}, \quad 1 \leq \sigma \leq m+1, \sigma \neq L,$$

and the recurrence formula

$$a_1 = \frac{\bar{y}_1^L}{y_1^L}, \quad a_l = \frac{1}{y_1^L}\left(\bar{y}_l^L - \sum_{p=2}^{l} y_p^L \sum_{(I_1, I_2, \dots, I_p)} a_{|I_1|} a_{|I_2|} \dots a_{|I_p|}\right).$$

PROOF. This assertion is a direct consequence of the group action of L^r on $\operatorname{Imm} T^r\mathbb{R}^{m+1}$, expressed in coordinates by Lemma 2. □

Now, using Lemma 3, we construct new coordinate functions on each of the sets $V^{r,L}$, $L = 1, 2, \dots, m+1$, adapted to the canonical action of L^r on $\operatorname{Imm} T^r\mathbb{R}^{m+1}$.

Theorem 2. *Let* y^K *be the canonical coordinates on a chart on* $\mathbb{R}^{m+1}$. *Fix an index* L, $1 \leq L \leq m+1$, *and let* σ *be an index, belonging to the complementary sequence of the index* L.

(a) *There exist unique functions $w^\sigma, w^\sigma_1, w^\sigma_2, \dots, w^\sigma_r$, defined on $V^{r,L}$ such that*

$$y^\sigma = w^\sigma, \quad y^\sigma_l = \sum_{p=1}^{l} \sum_{(I_1, I_2, \dots, I_p)} y^L_{|I_1|} y^L_{|I_2|} \dots y^L_{|I_p|} w^\sigma_p. \tag{14}$$

These functions are L^r-invariant.

(b) *The functions $w^L, w^L_1, w^L_2, \dots, w^L_r, w^\sigma, w^\sigma_1, w^\sigma_2, \dots, w^\sigma_r$, where*

$$w^L = y^L, \quad w^L_1 = y^L_1, \quad w^L_2 = y^L_2, \dots, \quad w^L_r = y^L_r, \tag{15}$$

are coordinate functions on $V^{r,L}$.

(c) *The canonical group action of the differential group L^r on* $\mathrm{Imm}\, T^r\mathbb{R}^{m+1}$ *is expressed by the equations*

$$\bar{w}^L = w^L, \quad \bar{w}^\sigma = w^\sigma, \quad \bar{w}^\sigma_l = w^\sigma_l,$$

$$\bar{w}^L_l = \sum_{p=1}^{l} w^L_p \sum_{(I_1, I_2, \dots, I_p)} a_{|I_1|} a_{|I_2|} \dots a_{|I_p|}.$$

Equations of the L^r-orbits are

$$w^L = c^L, \quad w^\sigma = c^\sigma, \quad w^\sigma_l = c^\sigma_l,$$

where $c^L, c^\sigma, c^\sigma_l \in \mathbb{R}$.

Proof. The proof that the system of algebraic equations (14) can be solved with respect to the functions $w^\sigma, w^\sigma_1, w^\sigma_2, \dots, w^\sigma_r$, is based on combinatorial constructions. □

3. Higher order homogeneous functions

Let r be a positive integer, and let us consider the manifold of regular velocities $\mathrm{Imm}\, T^r\mathbb{R}^{m+1}$ endowed with the canonical coordinates $y^K, y^K_1, y^K_2, \dots, y^K_r$. Suppose we have a function $F : \mathrm{Imm}\, T^r\mathbb{R}^{m+1} \to \mathbb{R}$. Let I be an open interval, and let $\gamma : I \to \mathbb{R}^{m+1}$ be an immersion. Any compact subinterval S of I associates with F the integral

$$F_S(\gamma) = \int_S (F \circ T^r\gamma)(t)dt. \tag{16}$$

In the following theorem we use the canonical right action of the differential group L^r on $\mathrm{Imm}\, T^r\mathbb{R}^{m+1}$, characterized by Lemma 2. We denote by $a_1(A)$ the first canonical coordinate of an element $A = J^r_0\alpha$ of the differential group L^r.

Theorem 3. *Let $\gamma : I \to \mathbb{R}^{m+1}$ be an immersion, J an open interval, and $\mu : J \to I$ a diffeomorphism such that $D\mu(s) > 0$ on J. The following conditions are equivalent:*

(a) *For any two compact intervals $J_0 \subset J$ and $I_0 \subset I$ such that $\mu(J_0) = I_0$,*

$$F_{I_0}(\gamma) = F_{J_0}(\gamma \circ \mu). \tag{17}$$

(b) *The function F satisfies on I*

$$(F \circ T^r \gamma)(t) = (F \circ T^r(\gamma \circ \mu))(\mu^{-1}(t)) D\mu^{-1}(t). \tag{18}$$

(c) *The function F satisfies*

$$F(P \circ A) = a_1(A) \cdot F(P) \tag{19}$$

for all $P \in \operatorname{Imm} T^r \mathbb{R}^{m+1}$ and $A \in L^r$.

Proof. 1. We prove that (a) is equivalent with (b). The transformation formula for an integral yields, because $D\mu(s) > 0$,

$$F_{I_0}(\gamma) = \int_{I_0} (F \circ T^r \gamma)(t) dt = \int_{J_0} (F \circ T^r \gamma)(\mu(s)) D\mu(s) ds, \tag{20}$$

and since

$$F_{J_0}(\gamma \circ \mu) = \int_{J_0} F(T^r(\gamma \circ \mu))(s) ds \tag{21}$$

for every J_0, condition (a) implies $(F \circ T^r \gamma)(\mu(s)) D\mu(s) = F(T^r(\gamma \circ \mu))(s)$ on J. Setting $\mu(s) = t$ we get (b). The converse is obvious.

2. We prove equivalence of (b) and (c). Suppose that (b) holds. From (18) and Theorem 1, (13),

$$\begin{aligned} F(T^r \gamma(t)) &= F(T^r(\gamma \circ \mu)(\mu^{-1}(t))) D\mu^{-1}(t) \\ &= F(T^r \gamma(\mu(\mu^{-1}(t))) \circ \mu^r(\mu^{-1}(t))) D\mu^{-1}(t) \\ &= F(T^r \gamma(t) \circ \mu^r(\mu^{-1}(t))) D\mu^{-1}(t). \end{aligned} \tag{22}$$

Note that by (10) $T^r \gamma(t) = J^r_0(\gamma \circ \mathrm{tr}_{-t})$ and by (11) $\mu^r(\mu^{-1}(t)) = J^r_0 \mu_{\mu^{-1}(t)}$, hence

$$\mu_{\mu^{-1}(t)} = \mathrm{tr}_{\mu(\mu^{-1}(t))} \circ \mu \circ \mathrm{tr}_{-\mu^{-1}(t)} = \mathrm{tr}_t \circ \mu \circ \mathrm{tr}_{-\mu^{-1}(t)}$$

and

$$\mu^r(\mu^{-1}(t)) = J^r_0(\mathrm{tr}_t \circ \mu \circ \mathrm{tr}_{-\mu^{-1}(t)}). \tag{23}$$

From (23) we have

$$a_1(\mu^r(\mu^{-1}(t))) = D(\mathrm{tr}_t \circ \mu \circ \mathrm{tr}_{-\mu^{-1}(t)})(0) = D\mu(\mu^{-1}(t)). \tag{24}$$

Collecting these formulas together, we can rewrite (22) as

$$\begin{aligned} &F(J_0^r(\gamma \circ \mathrm{tr}_{-t})) \\ &\quad = F(J_0^r(\gamma \circ \mathrm{tr}_{-t}) \circ J_0^r(\mathrm{tr}_t \circ \mu \circ \mathrm{tr}_{-\mu^{-1}(t)})) \cdot a_1(J_0^r(\mathrm{tr}_t \circ \mu \circ \mathrm{tr}_{-\mu^{-1}(t)}))^{-1}. \end{aligned} \tag{25}$$

Setting

$$P = J_0^r(\gamma \circ \mathrm{tr}_{-t}), \quad A = J_0^r(\mathrm{tr}_t \circ \mu \circ \mathrm{tr}_{-\mu^{-1}(t)}), \tag{26}$$

we get $F(P) = F(P \circ A) \cdot a_1(A)^{-1}$. But for any given P and A, equations (26) can always be solved with respect to γ and μ. This proves that (b) implies (c). The converse is obvious. □

Using equations of the action of the differential group L^r on the manifold of velocities $\mathrm{Imm}\, T^r\mathbb{R}^{m+1}$ (Lemma 2), one can easily restate condition (19) in an explicit form. Denoting

$$\begin{aligned} \bar{y}^K &= y^K, \\ \bar{y}_1^K &= a_1 y_1^K, \\ \bar{y}_2^K &= a_1^2 y_2^K + a_2 y_1^K, \\ \bar{y}_3^K &= a_1^3 y_3^K + 3a_1 a_2 y_2^K + a_3 y_1^K, \\ &\dots \\ \bar{y}_{r-1}^K &= \sum_{p=1}^{r-1} y_p^K \sum_{(I_1, I_2, \dots, I_p)} a_{|I_1|} a_{|I_2|} \dots a_{|I_p|}, \\ \bar{y}_r^K &= \sum_{p=1}^{r} y_p^K \sum_{(I_1, I_2, \dots, I_p)} a_{|I_1|} a_{|I_2|} \dots a_{|I_p|}, \end{aligned} \tag{27}$$

we see that condition (19) is equivalent to saying that

$$F(\bar{y}^K, \bar{y}_1^K, \bar{y}_2^K, \dots, \bar{y}_r^K) = a_1 F(y^K, y_1^K, y_2^K, \dots, y_r^K) \tag{28}$$

for all points $(y^K, y_1^K, y_2^K, \dots, y_r^K) \in \mathrm{Imm}\, T^r\mathbb{R}^{m+1}$ and all real numbers $a_1, a_2, \dots, a_r$ such that $a_1 > 0$.

Condition (19), or (28), is called *positive homogeneity* condition. A function F, satisfying this condition, is said to be *positive homogeneous in the variables* $y_1^K, y_2^K, \dots, y_r^K$, or just *positive homogeneous* (of degree 1).

4. The Zermelo conditions

In this section we show that a necessary and sufficient condition for the function F defined on $\operatorname{Imm} T^r\mathbb{R}^{m+1}$ to be positive homogeneous in its derivative variables, is that this function satisfies the Zermelo conditions.

Theorem 4 (Euler–Zermelo theorem). *Let $F = F(y^K, y_1^K, y_2^K, \dots, y_r^K)$ be a function. The following two conditions are equivalent:*

(a) *F is positive homogeneous in the variables $y_1^K, y_2^K, \dots, y_r^K$.*

(b) *F satisfies the Zermelo conditions*

$$\frac{\partial F}{\partial y_1^K} y_1^K + 2\frac{\partial F}{\partial y_2^K} y_2^K + 3\frac{\partial F}{\partial y_3^K} y_3^K + \dots + r\frac{\partial F}{\partial y_r^K} y_r^K = F, \tag{29}$$

and

$$\frac{\partial F}{\partial y_{r-k+1}^K} y_1^K + \binom{r-k+2}{1}\frac{\partial F}{\partial y_{r-k+2}^K} y_2^K + \binom{r-k+3}{2}\frac{\partial F}{\partial y_{r-k+3}^K} y_3^K + \dots + \binom{r}{k-1}\frac{\partial F}{\partial y_r^K} y_k^K = 0, \quad k = 1, 2, \dots, r-1. \tag{30}$$

Proof. 1. Suppose that F satisfies condition (a). Then differentiating the identity (28) with respect to the variables $a_1, a_2, a_3, \dots, a_r$ at the identity element $(1, 0, 0, \dots, 0)$ of the group L^r, we get (b).

2. To prove the converse, suppose that we have a function F, satisfying conditions (29) and (30). We define a new function G with the help of the transformation formulas (14) and (15) of Theorem 2 by

$$G(w^L, w_1^L, w_2^L, \dots, w_r^L, w^\sigma, w_1^\sigma, w_2^\sigma, \dots, w_r^\sigma) = F(y^K, y_1^K, y_2^K, \dots, y_r^K). \tag{31}$$

Applying the group action of the differential group L^r on both sides, we get the identity

$$G(\bar{w}^L, \bar{w}_1^L, \bar{w}_2^L, \dots, \bar{w}_r^L, \bar{w}^\sigma, \bar{w}_1^\sigma, \bar{w}_2^\sigma, \dots, \bar{w}_r^\sigma) = F(\bar{y}^K, \bar{y}_1^K, \bar{y}_2^K, \dots, \bar{y}_r^K), \tag{32}$$

where by Lemma 2

$$\bar{y}^K = y^K, \quad \bar{y}_l^K = \sum_{p=1}^{l} y_p^K \sum_{(I_1, I_2, \dots, I_p)} a_{|I_1|} a_{|I_2|} \dots a_{|I_p|}, \tag{33}$$

and by Theorem 2, (c),

$$\bar{w}^L = w^L, \quad \bar{w}^\sigma = w^\sigma, \quad \bar{w}^\sigma_l = w^\sigma_l,$$

$$\bar{w}^L_l = \sum_{p=1}^{l} w^L_p \sum_{(I_1,I_2,\ldots,I_p)} a_{|I_1|}a_{|I_2|}\ldots a_{|I_p|}. \tag{34}$$

Differentiating (32) with respect to a_l, $1 \le l \le r$, at the identity element $(1,0,0,\ldots,0)$ of L^r we get

$$\begin{aligned}\frac{\partial G}{\partial w^L_1}w^L_1 + 2\frac{\partial G}{\partial w^L_2}w^L_2 + 3\frac{\partial G}{\partial w^L_3}w^L_3 + \cdots + r\frac{\partial G}{\partial w^L_r}w^L_r \\ = \frac{\partial F}{\partial y^K_1}y^K_1 + 2\frac{\partial F}{\partial y^K_2}y^K_2 + 3\frac{\partial F}{\partial y^K_3}y^K_3 + \cdots + r\frac{\partial F}{\partial y^K_r}y^K_r,\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial G}{\partial w^L_{r-k+1}}w^L_1 + \binom{r-k+2}{1}\frac{\partial G}{\partial w^L_{r-k+2}}w^L_2 + \binom{r-k+3}{2}\frac{\partial G}{\partial w^L_{r-k+3}}w^L_3 \\ &\qquad + \cdots + \binom{r}{k-1}\frac{\partial G}{\partial w^L_r}w^L_k \\ &= \frac{\partial F}{\partial y^K_{r-k+1}}y^K_1 + \binom{r-k+2}{1}\frac{\partial F}{\partial y^K_{r-k+2}}y^K_2 + \binom{r-k+3}{2}\frac{\partial F}{\partial y^K_{r-k+3}}y^K_3 \\ &\qquad + \cdots + \binom{r}{k-1}\frac{\partial F}{\partial y^K_r}y^K_k, \quad k = 1,2,\ldots,r-1\end{aligned}$$

(no summation through L). Hence from (29) and (30)

$$\frac{\partial G}{\partial w^L_1}w^L_1 + 2\frac{\partial G}{\partial w^L_2}w^L_2 + 3\frac{\partial G}{\partial w^L_3}w^L_3 + \cdots + r\frac{\partial G}{\partial w^L_r}w^L_r = G, \tag{35}$$

and

$$\begin{aligned}&\frac{\partial G}{\partial w^L_{r-k+1}}w^L_1 + \binom{r-k+2}{1}\frac{\partial G}{\partial w^L_{r-k+2}}w^L_2 + \binom{r-k+3}{2}\frac{\partial G}{\partial w^L_{r-k+3}}w^L_3 \\ &\qquad + \cdots + \binom{r}{k-1}\frac{\partial G}{\partial w^L_r}w^L_k = 0, \quad k = 1,2,\ldots,r-1.\end{aligned} \tag{36}$$

But since $w^L_1 \neq 0$, equations (36) and (35) now imply

$$\frac{\partial G}{\partial w^L_k} = 0, \quad k = 2,3,\ldots,r, \tag{37}$$

and

$$\frac{\partial G}{\partial w_1^L} w_1^L = G. \tag{38}$$

We can use these equations to complete the proof. From (38) and the classical inverse Euler theorem on homogeneous functions we find that the function G must be positive homogeneous of degree 1 in the variable w_1^L (see e.g. E. Lindelöf [11]), that is,

$$\begin{aligned} &G(w^L, a_1 w_1^L, w_2^L, \dots, w_r^L, w^\sigma, w_1^\sigma, w_2^\sigma, \dots, w_r^\sigma) \\ &\qquad = a_1 G(w^L, w_1^L, w_2^L, \dots, w_r^L, w^\sigma, w_1^\sigma, w_2^\sigma, \dots, w_r^\sigma) \end{aligned} \tag{39}$$

for all $a_1 > 0$. However, from (37), G does not depend on $w_2^L, w_3^L, \dots, w_r^L$. Thus, the left-hand side of (39) can be expressed as

$$\begin{aligned} &G(w^L, a_1 w_1^L, w_2^L, \dots, w_r^L, w^\sigma, w_1^\sigma, w_2^\sigma, \dots, w_r^\sigma) \\ &= G(\bar{w}^L, \bar{w}_1^L, \bar{w}_2^L, \dots, \bar{w}_r^L, \bar{w}^\sigma, \bar{w}_1^\sigma, \bar{w}_2^\sigma, \dots, \bar{w}_r^\sigma) = F(\bar{y}^K, \bar{y}_1^K, \bar{y}_2^K, \dots, \bar{y}_r^K), \end{aligned} \tag{40}$$

with arguments of the function G defined by equations (34). But the right-hand side of formula (39) is equal to $a_1 F(y^K, y_1^K, y_2^K, \dots, y_r^K)$, thus, assertion (a) is proved. □

The principal meaning of Theorem 4 consists in a characterization of higher order Lagrangians F whose extremals are set-solutions, and also general differential equations whose solutions are set-solutions. From this point of view the functions F can serve as fundamental functions for possible higher order generalizations of Finsler geometry. Clearly, the Finsler Lagrangians could be further specified by additional conditions.

One can also consider auxiliary Lagrangians of the form F^2, with similar use for finding extremals as in the first order case.

5. Homogeneous differential equations

Let r be a positive integer. Consider the manifold of regular velocities $\operatorname{Imm} T^r \mathbb{R}^{m+1}$ with the canonical coordinates $y^K, y_1^K, y_2^K, \dots, y_r^K$, and a finite system of differential equations of order r

$$F_j(y^K, y_1^K, y_2^K, \dots, y_r^K) = 0 \tag{41}$$

for unknown *regular* curves $\gamma(t) = (y^1(t), y^2(t), \dots, y^{m+1}(t))$ in $\mathbb{R}^{m+1}$, i.e., for immersions of open intervals in the real line $\mathbb{R}$ into $\mathbb{R}^{m+1}$. We wish to find conditions under which for every solution γ of the system (41), the curve $\gamma \circ \tau$, obtained from γ by a reparametrization τ of its domain of definition, is again a solution of (41).

More formally, we say that a solution $\gamma : I \to \mathbb{R}^{m+1}$ of the system (41), defined on an open interval $I \subset \mathbb{R}$, is a *set-solution*, if for every diffeomorphism of open intervals $\tau : J \to I$, the curve $\gamma \circ \tau : J \to \mathbb{R}^{m+1}$ is again a solution.

Theorem 5 (Set-solutions)**.** *Suppose that a system of differential equations*

$$F_j(y^K, y_1^K, y_2^K, \dots, y_r^K) = 0$$

is defined by positive homogeneous functions $F_j : \mathrm{Imm}\, T^r\mathbb{R}^{m+1} \to \mathbb{R}$. *Then every solution of this system is a set-solution.*

Proof. From Theorem 1, (b), the r-jet prolongation $T^r\gamma$ satisfies

$$T^r(\gamma \circ \mu)(s) = T^r\gamma(\mu(s)) \circ \mu^r(s).$$

Then, however, from Theorem 3, (c), the positive homogeneity of F_j means that on J,

$$F_j(T^r(\gamma \circ \mu)(s)) = F_j(T^r\gamma(\mu(s)) \circ \mu^r(s)) = a_1(\mu^r(s)) \cdot F_j(T^r\gamma(\mu(s))),$$

or, which is the same, on I,

$$F_j(T^r(\gamma \circ \mu)(\mu^{-1}(t))) = a_1(\mu^r(\mu^{-1}(t))) \cdot F_j(T^r\gamma(t)).$$

Consequently, $F_j(T^r\gamma(t)) = 0$ if and only if $F_j(T^r(\gamma \circ \mu)(s)) = 0$. □

A system of differential equations is said to be *positive homogeneous*, if it is expressible in the form (41) with positive homogeneous functions F_j.

Remark. An example of a system of second order differential equations and its analysis from the geometric and variational point of view will be an objective of our subsequent paper [18].

On the other hand, examples of second order positive homogeneous equations in the standard sense (geodesic equations) appear in Riemann and Finsler geometry (see e.g. Bao, Chern and Shen [1], Miron [15]). It seems that a direct way how to replace these equations by those homogeneous in Zermelo sense has not been completely understood yet, and it is still subject of further research.

6. Variational equations

We conclude this paper with the discussion of a class of second order *variational* differential equations. To this purpose we first present, in a slightly simplified version, basic definitions.

Consider the manifolds of velocities $\operatorname{Imm} T^1\mathbb{R}^{m+1}$ and $\operatorname{Imm} T^2\mathbb{R}^{m+1}$ with canonical coordinates $y^K, \dot{y}^K$ and $y^K, \dot{y}^K, \ddot{y}^K$, and a function $F : \operatorname{Imm} T^1\mathbb{R}^{m+1} \to \mathbb{R}$. Let I be an open interval, and let $\gamma : I \to \mathbb{R}^{m+1}$ be an immersion. Any compact subinterval S of I associates with F the integral

$$F_S(\gamma) = \int_S (F \circ T^1\gamma)(t)dt, \tag{42}$$

defining, in a well-known sense, the *variational functional* $\gamma \to F_S(\gamma)$. The *Euler–Lagrange expressions* associated with F are real functions $E_K(F)$, defined on $\operatorname{Imm} T\mathbb{R}^{m+1}$ by

$$E_K(F) = \frac{\partial F}{\partial y^K} - \frac{\partial^2 F}{\partial y^L \partial \dot{y}^K}\dot{y}^L - \frac{\partial^2 F}{\partial \dot{y}^L \partial \dot{y}^K}\ddot{y}^L,$$

and the differential equations for extremals of the variational functional (42) are

$$E_K(F) = 0.$$

Now consider a system of functions ε_L, defined on $\operatorname{Imm} T^2\mathbb{R}^{m+1}$, where $L = 1, 2, \dots, m+1$. We say that this system is *variational*, if there exists a function $F : \operatorname{Imm} T^1\mathbb{R}^{m+1} \to \mathbb{R}$ such that

$$\varepsilon_L = E_L(F),$$

i.e., the functions ε_L coincide with the Euler–Lagrange expressions of F. If F exists, we call it the *Lagrangian* for the system $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m+1}\}$. Clearly, for variational systems $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m+1}\}$, the corresponding second order differential equations

$$\varepsilon_L(y^K, \dot{y}^K, \ddot{y}^K) = 0$$

coincide with equations for extremals of a certain variational functional (42).

In the following theorem, the concept of positive homogeneity is used in the sense as introduced in this paper; for first order functions (Lagrangians) this concept coincides with the classical one.

Theorem 6. *Suppose that the system of functions* $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m+1}\}$ *is variational. Then* $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m+1}\}$ *admits a positive homogeneous Lagrangian if and only if the functions* $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m+1}$ *are positive homogeneous.*

Proof. 1. It is immediately seen by a direct computation that if F is a positively homogeneous function, then the functions $E_K(F)$ are positively homogeneous, i.e., satisfy

$$\frac{\partial E_K(F)}{\partial \dot{y}^M}\dot{y}^M + 2\frac{\partial E_K(F)}{\partial \ddot{y}^M}\ddot{y}^M = E_K(F), \quad \frac{\partial E_K(F)}{\partial \ddot{y}^M}\dot{y}^M = 0 \tag{43}$$

(Theorem 4, $r = 2$). Indeed, if

$$F = \frac{\partial F}{\partial \dot{y}^M}\dot{y}^M, \tag{44}$$

then we have

$$\begin{aligned}\frac{\partial E_K(F)}{\partial \dot{y}^M}\dot{y}^M + 2\frac{\partial E_K(F)}{\partial \ddot{y}^M}\ddot{y}^M &= \frac{\partial^2 F}{\partial \dot{y}^M \partial y^K}\dot{y}^M - \frac{\partial^3 F}{\partial \dot{y}^M \partial y^L \partial \dot{y}^K}\dot{y}^L\dot{y}^M \\ &\quad - \frac{\partial^2 F}{\partial y^M \partial \dot{y}^K}\dot{y}^M - \frac{\partial^3 F}{\partial \dot{y}^M \partial \dot{y}^L \partial \dot{y}^K}\ddot{y}^L\dot{y}^M - 2\frac{\partial^2 F}{\partial \dot{y}^M \partial \dot{y}^K}\ddot{y}^M,\end{aligned}$$

and

$$\begin{aligned}E_K(F) &= \frac{\partial}{\partial y^K}\left(\frac{\partial F}{\partial \dot{y}^M}\dot{y}^M\right) - \frac{\partial^2}{\partial y^L \partial \dot{y}^K}\left(\frac{\partial F}{\partial \dot{y}^M}\dot{y}^M\right)\dot{y}^L - \frac{\partial^2}{\partial \dot{y}^L \partial \dot{y}^K}\left(\frac{\partial F}{\partial \dot{y}^M}\dot{y}^M\right)\ddot{y}^L \\ &= \frac{\partial^2 F}{\partial \dot{y}^M \partial y^K}\dot{y}^M - \frac{\partial^3 F}{\partial y^L \partial \dot{y}^K \partial \dot{y}^M}\dot{y}^M\dot{y}^L - \frac{\partial^2 F}{\partial y^L \partial \dot{y}^K}\dot{y}^L \\ &\quad - \frac{\partial^3 F}{\partial \dot{y}^L \partial \dot{y}^K \partial \dot{y}^M}\dot{y}^M\ddot{y}^L - \frac{\partial^2 F}{\partial \dot{y}^K \partial \dot{y}^L}\ddot{y}^L - \frac{\partial^2 F}{\partial \dot{y}^L \partial \dot{y}^K}\ddot{y}^L,\end{aligned}$$

proving the first identity (43). The second identity also follows from (44):

$$\frac{\partial E_K(F)}{\partial \ddot{y}^M}\dot{y}^M = -\frac{\partial^2 F}{\partial \dot{y}^M \partial \dot{y}^K}\dot{y}^M = 0.$$

2. Conversely, assume that the functions ε_K are positive homogeneous. Hence, by Theorem 4,

$$\frac{\partial \varepsilon_K}{\partial \dot{y}^M}\dot{y}^M + 2\frac{\partial \varepsilon_K}{\partial \ddot{y}^M}\ddot{y}^M = \varepsilon_K, \qquad \frac{\partial \varepsilon_K}{\partial \ddot{y}^M}\dot{y}^M = 0. \tag{45}$$

Denoting F the Lagrangian for the system $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m+1}\}$, we have

$$0 = \frac{\partial \varepsilon_K}{\partial \ddot{y}^M}\dot{y}^M = -\frac{\partial^2 F}{\partial \dot{y}^M \partial \dot{y}^K}\dot{y}^M = -\frac{\partial}{\partial \dot{y}^K}\left(\frac{\partial F}{\partial \dot{y}^M}\dot{y}^M - F\right). \tag{46}$$

Using (46), the first equation of (45) gives rise to two independent conditions

$$\frac{\partial^2}{\partial \dot{y}^L \partial \dot{y}^K}\left(\frac{\partial F}{\partial \dot{y}^M}\dot{y}^M\right) = \frac{\partial^2 F}{\partial \dot{y}^L \partial \dot{y}^K}, \tag{47}$$

$$\frac{\partial}{\partial y^K}\left(\frac{\partial F}{\partial \dot{y}^M}\dot{y}^M - F\right) = 0, \tag{48}$$

where (47) is a consequence of (46). From (46) and (48), it follows that the function

$$\frac{\partial F}{\partial \dot{y}^M}\dot{y}^M - F$$

is constant on $\operatorname{Imm} T^1\mathbb{R}^{m+1}$, for instance, equal $c \in \mathbb{R}$. Then the function $F + c$ is the positive homogeneous Lagrangian for the system $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m+1}\}$. This completes the proof. □

Theorem 6, namely the identities (43), complete the classical understanding of equations for extremals of parameter-invariant integral variational problems (cf. I. M. Gelfand and S. V. Fomin [5]).

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