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On homogeneous submanifolds of negatively curved Riemannian manifolds

By REZA MIRZAIE (Qazvin)

Abstract. We give a description of the orbits of some isometric actions on Riemannian manifolds of negative curvature.

1. Introduction

The authors of [4] gave a description of homogeneous submanifolds of the Hyperbolic space $H^n(c)$, $c < 0$. Among other results they proved: *If G is a connected subgroup of the isometries of $H^n(c)$ and the fixed point set of the action of G on M is empty, then either there is a geodesic orbit or all orbits are included in horospheres centered at the same point at infinity (so there is a class $[\gamma]$ of asymptotic geodesics such that $G[\gamma] = [\gamma]$).* A similar result is true if G is a connected and solvable subgroup of the isometries of a simply connected Riemannian manifold M of negative curvature. We use this result as a tool to study topological properties of some cohomogeneity two Riemannian manifolds of negative curvature. We recall that if G is a closed and connected subgroup of the isometries of a Riemannian manifold M, the number $\dim M - \max_{x\in M} \dim G(x)$ is called the cohomogeneity of the action of G on M. When the cohomogeneity is small, we expect close geometrical and topological relations between M, G and G-orbits of M. If M has negative curvature and the cohomogeneity is zero (M is a homogeneous G-manifold), S. Kobayashi proved that M must be simply connected [10].

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If M is negatively curved and the cohomogeneity is one, it is proved that either M is simply connected or the fundamental group of M is isomorphic to Z^p for some positive integer p. In the later case, if $p > 1$ then each orbit is diffeomorphic to $R^{n-1-p} \times T^p$, $n = \dim M$, and M is diffeomorphic to $R^{n-p} \times T^p$. If $p = 1$ then there is an orbit diffeomorphic to S^1 and the other orbits are covered by $S^{n-2} \times R$ [15].

There is no complete classification result on cohomogeneity two Riemanniam manifolds of negative curvature. But there are some results under conditions on G and curvature of M. Let $\mathrm{Fix}(G, M) = \{x \in M : G(x) = x\}$. If $\mathrm{Fix}(G, M) \neq \emptyset$ and M is negatively curved and of cohomogeneity two under the action of G, then M is diffeomorphic to $S^1 \times R^{n+1}$ or $B^2 \times R^n$ (B^2 is the mobius band), and the principal orbits are diffeomorphic to S^n [12]. Also we have studied cohomogeneity two Riemannian manifolds of constant negative curvature [13]. In Theorem 3.3 of the present paper, we study cohomogeneity two Riemannian G-manifolds of negative curvature under the conditions that the singular orbits (if there is any) are the fixed points of G and G is non-semisimple.

2. Preliminaries

First we mention some definitions and facts which we will use in the proofs. If M is a Riemannian manifold, we denote by $\mathrm{Iso}(M)$ the Lie group of all isometries of M. If $\delta \in \mathrm{Iso}(M)$, the squared displacement function $d_\delta^2 : M \to M$ is defined by

$$d_\delta^2(x) = d(x, \delta x)$$

Fact 2.1 (See [1]). *If M is a simply connected Riemannian manifold of negative curvature and $\delta \in \mathrm{Iso}(M)$, then one of the followings is true:*

(1) *d_δ^2 has no minimum point.*

(2) *Minimum point set of d_δ^2 is equal to the fixed point set of δ.*

(3) *Minimum point set of d_δ^2 is the image of a geodesic γ translated by δ (i.e., there is a positive number t_0 such that for all t, $\delta(\gamma(t)) = \gamma(t + t_0)$).*

Isometries (1), (2), and (3) are called parabolic, elliptic, and axial, respectively.

We recall (see [8]) that infinity $M(\infty)$ of a simply connected Riemannian manifold M of non-positive curvature is the classes of asymptotic geodesics. For each geodesic γ we denote by $[\gamma]$ the asymptotic class of geodesics containing γ. If $x \in M$ then there is a unique (up to parametrization) geodesic γ_x in the class $[\gamma]$ containing x, and there is a unique hypersurface S_x containing x and perpendicular to all elements of $[\gamma]$. S_x is called a horosphere related to $[\gamma]$.

Fact 2.2. *Let M be a simply connected Riemannian manifold of negative curvature.*

a) *If g is an axial isometry of M then the geodesic γ with the property $g(\gamma) = \gamma$ is unique.*

b) *If g is a parabolic isometry of M then there is a unique class of asymptotic geodesics $[\gamma]$ such that $g[\gamma] = [\gamma]$.*

Proof. (a) is a direct consequence of Proposition 4.2(3) in [1].

(b) By Lemma 6.1 in [7], g has a fixed point in $M(\infty)$, so there is a class $[\gamma]$ of asymptotic geodesics such that $g[\gamma] = [\gamma]$, and by Proposition 6.4 in [7], $[\gamma]$ is unique (because if not, g must be elliptic or axial). □

Fact 2.3. *Let G be a connected and solvable Lie subgroup of isometries of a simply connected and negatively curved Riemannian manifold M. Then one of the followings is true:*

(1) $\mathrm{Fix}(G, M) \neq \emptyset$.

(2) *There is a unique G-invariant geodesic.*

(3) *There is a unique class of asymptotic geodesics $[\gamma]$ such that $G[\gamma] = [\gamma]$.*

Proof. For the proof of existence, see Theorem 5 in [3]. Uniqueness in (2), (3) comes from Fact 2.2. □

Corollary 2.4. *If M is a simply connected Riemannian manifold of negative curvature and G is a closed and connected subgroup of* $\mathrm{Iso}(M)$ *such that* $\mathrm{Fix}(G, M) = \emptyset$, *then there is at most one totally geodesic G-orbit in M.*

Proof. The proof of this corollary is as like as the proof of Lemma 3.1 of [4] which we rewrite it for facility. Denote by $\overline{\nabla}$ and ∇ the Riemannian connections of M and submanifolds of M. Suppose that $G(q'), G(q)$ are distinct totally geodesic orbits of M. Consider a point $p \in G(q')$ such that $d(q, G(q')) = d(q, p)$. Let γ be a minimizing geodesic such that $\gamma(0) = p, \gamma(1) = q$. Then, $\gamma'(0)$ is perpendicular to $G(q')(= G(p))$ at the point p. If N is a G-orbit and $a \in N$, then the tangent space T_aN is generated by

$$\{Y(a) : Y \text{ is a vector field in the Lie algebra of } G\}.$$

Consider a vector field Y in the Lie algebra of G and put $g(t) = \langle Y(\gamma(t)), \gamma'(t)\rangle$. Then

$$g'(t) = \frac{d}{dt}\langle Y(\gamma(t)), \gamma'(t)\rangle = \langle \overline{\nabla}_{\gamma'(t)} Y, \gamma'(t)\rangle$$

Since Y is a Killing vector field (see [14], p. 255) then $g'(t) = 0$. Since $\gamma'(0)$ is perpendicular to $G(p)$ then $g(0) = 0$, so for each $t \in I$, $g(t) = 0$. Then for each $t \in I$, $\gamma'(t)$ is perpendicular to $G(\gamma(t))$. Since $G(q) \neq q$, there is a vector field X in the Lie algebra of G such that $X(q) \neq 0$. Put

$$f(t) = -\langle S_{\gamma'(t)}(X(\gamma(t)), X(\gamma(t))\rangle$$

Where $S_{\gamma'(t)}$ is the shape operator of $G(\gamma(t))$. $G(p)$ and $G(q)$ are totally geodesic, then

$$f(0) = f(1) = 0 \tag{$*$}$$

The vector field $X(\gamma(t))$ is a Jaccobi vector field along γ (see [14], p. 252, Lemma 26). Thus

$$X'' + R(\gamma', X)\gamma' = 0$$

and X is a Killing vector field, so

$$-\langle \overline{\nabla}_X X, \gamma'(t)\rangle = \langle \overline{\nabla}_{\gamma'} X, X\rangle$$

Then we have:

$$\begin{aligned}
f(t) = -\langle \overline{\nabla}_X X - \nabla_X X, \gamma'(t)\rangle = -\langle \overline{\nabla}_X X, \gamma'(t)\rangle = \langle \overline{\nabla}_{\gamma'} X, X\rangle \\
\Rightarrow f'(t) = \frac{d}{dt}\langle \overline{\nabla}_{\gamma'} X, X\rangle = \langle X'', X\rangle + \langle \overline{\nabla}_{\gamma'(t)} X, \overline{\nabla}_{\gamma'(t)} X\rangle \\
= -\langle R(\gamma'(t), X)\gamma'(t), X\rangle + \langle \overline{\nabla}_{\gamma'} X, \overline{\nabla}_{\gamma'} X\rangle
\end{aligned}$$

Since M is negatively curved then $f'(t) > 0$, which is a contradiction by $(*)$. □

Remark 2.5. If M is a Riemannian manifold and G is a connected subgroup of $\mathrm{Iso}(M)$, and if $\widetilde{M}$ is the universal Riemannian covering manifold of M with the covering map $\kappa : \widetilde{M} \to M$, then there is a connected covering $\widetilde{G}$ of G with the covering map $\pi : \widetilde{G} \to G$, such that $\widetilde{G}$ acts isometrically on $\widetilde{M}$ and

(1) Each deck transformation δ of the covering $\kappa : \widetilde{M} \to M$ maps $\widetilde{G}$-orbits on to $\widetilde{G}$-orbits.

(2) If $x \in M$ and $\widetilde{x} \in \widetilde{M}$ then $\kappa(\widetilde{G}(\widetilde{x})) = G(x)$.

(3) $\mathrm{Fix}(\widetilde{G}, \widetilde{M}) = \kappa^{-1}(\mathrm{Fix}(G, M))$.

(4) If G is non-semisimple then $\widetilde{G}$ is no-semisimple.

(5) The deck transformation group, which we denote it by Δ, centralizes $\widetilde{G}$ (i.e., for each $\delta \in \Delta$ and $\widetilde{g} \in \widetilde{G}$, $\delta\widetilde{g} = \widetilde{g}\delta$).

Proof. $\widetilde{G}$ can be defined in a similar way in [2] pages 63, 64. (1), (2), (3) and (4) are simple consequences of the definition of $\widetilde{G}$. The proof of (5) can be made as a similar way in the proof of Theorem 9.1 in [2]. □

Remark 2.6. Let $\widetilde{M}$ be a complete and simply connected Riemannian manifold of strictly negative curvature (curvature is $\leq c < 0$, for a constant number c) and let S be a horosphere in $\widetilde{M}$ related to asymptotic class of geodesics $[\gamma]$. The function $f : \widetilde{M} \to R, f(p) = \lim_{t\to\infty} d(p, \gamma(t)) - t$, is called a Bussmann function.

(a) For each point $p \in \widetilde{M}$ there is a point $\eta_S(p)$ in S, which is the unique point of S nearest p, and the following map is a homeomorphism:

$$\phi : \widetilde{M} \to S \times R, \quad \phi(p) = (\eta_S(p), f(p)).$$

(b) If g is an isometry of $\widetilde{M}$ such that $g[\gamma] = [\gamma]$ (g leaves invariant the horosphere foliation related to $[\gamma]$) then $gS = S$ or g is axial and the axes of g belongs to $[\gamma]$.

Proof. For (a) see [7], p. 57, 58, Propositions 3.2 and 3.4. Proof of (b) is as like as the proof of Lemma 3 in [3]. □

Lemma 2.7 (See [13]). *Let M be a Riemannian manifold of negative curvature, $n = \dim M \geq 3$, and $\widetilde{M}$ be its universal covering. If there is a geodesic γ on $\widetilde{M}$ and an element δ in the center of the deck transformation group Δ, such that $\delta\gamma = \gamma$, then M is diffeomorphic to one of the following spaces*

$$S^1 \times R^{n-1}, \quad B^2 \times R^{n-2}$$

where, B^2 is the mobius band.

3. Results

In the present section we study topological properties of some cohomogeneity two Riemannian manifolds of negative curvature. We refer to [2] and [11] for definitions and details about singular and principal orbits of the actions of Lie groups on manifolds.

Theorem 3.1 (See [13]). *Let M^{n+2} be a complete negatively curved and non-simply connected Riemannian manifold which is of cohomogeneity two under the action of a closed and connected Lie subgroup of isometries. If* $\mathrm{Fix}(G, M) \neq \emptyset$ *then*

(a) *M is diffeomorphic to $S^1 \times R^{n+1}$ or $B^2 \times R^n$ (B^2 is the mobius band).*

(b) $\mathrm{Fix}(G, M)$ *is diffeomorphic to S^1.*

(c) *Each principal orbit is diffeomorphic to S^n.*

Remark 3.2. By Theorem 3.7 (a) in [15], if M is a non-simply connected and complete Riemannian manifold which is of cohomogeneity one under the action of a connected and closed subgroup of isometries, and if there is not any singular orbit, then there are positive integers p, s such that M is diffeomorphic to $R^p \times R^{s+1}$ and each orbit is diffeomorphic to $R^p \times R^s$, $p + s = \dim M - 1$.

Theorem 3.3. *Let M^{n+2} be a complete Riemannian manifold of strictly negative curvature and let G be a closed, connected and non-semisimple subgroup of isometries of M^{n+2}. If M is a cohomogeneity two G-manifold such that the singular orbits (if there is any) are fixed points of G. Then one of the following is true:*

(1) *M is simply connected (diffeomorphic to R^{n+2}).*

(2) *M is diffeomorphic to $S^1 \times R^{n+1}$ or $B^2 \times R^n$ (B^2 is the mobius band). Each principal orbit is diffeomorphic to S^n. Union of singular orbits* ($\mathrm{Fix}(G, M)$) *is diffeomorphic to S^1.*

(3) *M is diffeomorphic to $S^1 \times R^2$ or $B^2 \times R$. All orbits are diffeomorphic to S^1.*

(4) *$\pi_1(M) = Z^p$ for some positive integer p, and all orbits are diffeomorphic to $R^{n-p} \times T^p$.*

(5) *M is a parabolic manifold homeomorphic to $M_1 \times R$. Where, M_1 is a cohomogeneity one G-manifold and there is a horosphere S in the universal Riemannian covering of M such that M_1 is diffeomorphic to $\frac{S}{\pi_1(M)}$.*

Proof. Following Remark 2.5, let $\widetilde{M}$ be the universal Riemannian covering manifold of M with the deck transformation group Δ and let $\widetilde{G}$ be the corresponding connected covering of G which acts isometrically and by cohomogeneity two on $\widetilde{M}$. If $\mathrm{Fix}(\widetilde{G}, \widetilde{M}) \neq \emptyset$ then $\mathrm{Fix}(G, M) \neq \emptyset$, so by Theorem 3.1, we get the parts (1) or (2) of the theorem. Now, we suppose that

$$\mathrm{Fix}(\widetilde{G}, \widetilde{M}) = \emptyset \qquad ((*))$$

By assumptions of the theorem, if there is a singualr orbit, it must be a fixed point, so by $(*)$ all $\widetilde{G}$-orbits in $\widetilde{M}$ must be n-dimensional. Since G is non-semisimple, $\widetilde{G}$ is non-semisimple. Let H be a solvable normal subgroup of $\widetilde{G}$ and put $N = \mathrm{Fix}(H, \widetilde{M})$. We consider following two cases separately:

(a) $N = \emptyset$ **(b)** $N \neq \emptyset$

(a): By Fact 2.3, one of the following is true:
(a-i) There is a unique geodesic γ such that $H(\gamma) = \gamma$.
(a-ii) There is a unique class of asymptotic geodesics $[\gamma]$ such that $H[\gamma] = [\gamma]$.
(a-i): From normality of H in $\widetilde{G}$ and uniqueness of γ, we get that $\widetilde{G}(\gamma) = \gamma$. Since $\mathrm{Fix}(\widetilde{G}, \widetilde{M}) = \emptyset$ then γ is a $\widetilde{G}$-orbit in $\widetilde{M}$. But all orbits are n-dimensional and the orbit γ is of dimension one. Thus all orbits are of dimension one and $n = 1$. Each $\delta \in \Delta$ maps $\widetilde{G}$-orbits onto $\widetilde{G}$-orbits. So $\delta(\gamma)$ is a $\widetilde{G}$-orbit. Since by Corollary 2.4, γ is the unique geodesic orbit, then $\delta(\gamma) = \gamma$. Thus $\Delta\gamma = \gamma$ and $\pi_1(M) = Z$ (see [6], Theorem 3.4 pa. 261). Now, by Lemma 2.7, M is diffeomorphic to $S^1 \times R^2$ or $B^2 \times R$. Since all G-orbits of M are regular (and diffeomorphic to each other) and the G-orbit $\frac{\gamma}{\Delta}$ is diffeomorphic to $\frac{\gamma}{Z} = \frac{R}{Z} = S^1$, all G-orbits are diffeomorphic to S^1. This is the part (3) of the theorem.
(a-ii) As like as (a-i), we get from normality of H in $\widetilde{M}$ and uniqueness of $[\gamma]$ that $\widetilde{G}[\gamma] = [\gamma]$. First, suppose that there is an axial element $\delta \in \Delta$ and let λ be the unique geodesic such that $\delta\lambda = \lambda$. If $g \in \widetilde{G}$, $\delta(g\lambda) = g\delta\lambda = g\lambda$. Then, we get from uniqueness of λ that $g\lambda = \lambda$. So, λ is a $\widetilde{G}$-orbit and we get part (3) of the theorem in the same way as (a-i). Now, suppose that all elements of Δ are non-axial. Since elements of Δ and $\widetilde{G}$ are commutative we get that $\Delta[\gamma] = [\gamma]$. Non-identity elements of Δ are fixed point free, so they are parabolic and M is a parabolic manifold. By Remark 2.6, for each $\delta \in \Delta$ and each horosphere S related to the asymptotic class $[\gamma]$, $\delta S = S$. Fix a horosphere S related to $[\gamma]$. Put $M_1 = \frac{S}{\Delta}$ and let η_S and f be the maps defined in Remark 2.6. The homeomorphism $\phi : \widetilde{M} \to S \times R$ mentioned in Remark 2.6, induces a homeomorphism $\phi_1 : \frac{\widetilde{M}}{\Delta} = M \to \frac{S}{\Delta} \times R = M_1 \times R$, such that $\phi_1(x) = (\kappa\eta_S(\widetilde{x}), f(\widetilde{x}))$, $\widetilde{x} \in \kappa^{-1}(x)$. Now, we show that for each $g \in \widetilde{G}$, $gS = S$. If $gS \neq S$ then we get from Remark 2.6, that g is axial isometry and there is a unique geodesic λ in $[\gamma]$ such that g translates it. Since the members of Δ and g are commutative, we get from uniqueness of λ that for each $\delta \in \Delta$, $\delta\lambda = \lambda$. But intersection of λ and S is a one point set. So, we get from $\delta S = S$ that δ has a fixed point, which is a contradiction for non-identity δ. Therefore, $gS = S$. This means that all $\widetilde{G}$-orbits of $\widetilde{M}$ are included in horospheres. Thus, S is a cohomogeneity one $\widetilde{G}$-manifold and $\frac{S}{\Delta}$ is a cohomogeneity one G-manifold. This is part (5) of the theorem.
(b): N is a nontrivial totally geodesic submanifold of $\widetilde{M}$. If $g \in \widetilde{G}$, $h \in H$ and $x \in N$ then

$$g^{-1}hg(x) = x \Rightarrow hg(x) = g(x) \Rightarrow g(x) \in N$$

Thus $\widetilde{G}(N) = N$. All orbits are of dimension n. So if $x \in N$ then

$$n = \dim \widetilde{G}(x) \leq \dim N < \dim \widetilde{M} = n + 2 \Rightarrow \dim N = n \text{ or } n + 1$$

Now, consider two cases $\dim N = n$ and $\dim N = n+1$ separately.
(b-j) $\dim N = n$.
In this case, N is a $\widetilde{G}$-orbit. If $n = 1$, in a similar way in (a-i) we get part (3) of the theorem. Suppose $n \geq 2$ and put $N_1 = \kappa(N)$. By Corollary 2.4, N is the unique totally geodesic $\widetilde{G}$-orbit in $\widetilde{M}$. Thus, for each $\delta \in \Delta$, $\delta(N) = N$, so $N_1 = \frac{N}{\Delta}$. But N_1 is a totally geodesic G-orbit in M, so it must be simply connected (since by Kobayashi's theorem in [10] homogeneous manifolds of negative curvature are simply connected). Therefore, Δ is trivial and M is simply connected. This is the part (1) of the theorem.
(b-jj) $\dim N = n+1$
Since all orbits are of dimension n, N is a negatively curved cohomogeneity one $\widetilde{G}$-manifold. Consider following two cases:
(b-jj-1): There is a $\delta \in \Delta$ and $x \in \widetilde{M}$ such that $\delta\widetilde{G}(x) \neq \widetilde{G}(x)$.
(b-jj-2): For each $\delta \in \Delta$ and $x \in \widetilde{M}$, $\delta\widetilde{G}(x) = \widetilde{G}(x)$.
(b-jj-1) From the fact that δ maps orbits on to orbits, we get that $\delta\widetilde{G}(x) = \widetilde{G}(y)$, $y \in \widetilde{M}$ (i.e., $\widetilde{G}(x) \cap \widetilde{G}(y) = \emptyset$). By Proposition 4.2 in [15], the minimum point set of the following function is at most the image of a geodesic

$$f_\delta : \widetilde{M} \to R, \quad f_\delta(x) = d^2(x, \delta(x))$$

So we can find a geodesic γ such that the image of γ is not the minimum point set of f_δ and $\gamma(0) \in G(x)$, $\gamma(1) \in G(y)$. Put $g(t) = f_\delta(\gamma(t))$. Since the elements of Δ and $\widetilde{G}$ are commutative, f_δ is constant along orbits (because $f_\delta(gx) = d^2(gx, \delta gx) = d^2(gx, g\delta x) = d^2(x, \delta x) = f_\delta(x)$). Since $\delta(\gamma(0)) \in G(\gamma(1))$, then $f_\delta(\delta\gamma(0)) = f_\delta(\gamma(1))$. Thus

$$\begin{aligned} g(0) &= f_\delta(\gamma(0)) = d^2(\gamma(0), \delta(\gamma(0))) \\ &= d^2(\delta(\gamma(0)), \delta^2(\gamma(0))) = f_\delta(\delta\gamma(0)) = f_\delta(\gamma(1)) = g(1) \end{aligned}$$

Since g is strictly convex (see [1]), it has a unique minimum point $t_0 \in (0,1)$. Therefore, $\widetilde{G}(\gamma(t_0))$ is the minimum point set of f_δ, which must be a geodesic. Then $\widetilde{G}(\gamma(t_0))$ is a (geodesic) one dimensional $\widetilde{G}$-orbit. Then in a similar way in $(a-i)$ we get part (3) of the theorem.
(b-jj-2): Put $N_1 = \kappa(N)$. Since for each $\delta \in \Delta$, $\delta(N) = N$ then $\pi_1(M) = \pi_1(N_1)$. N_1 is a cohomogeneity one G-manifold of negative curvature, without singular orbits. So, by Remark 3.2, each G-orbit in N_1 is diffeomorphic to $T^p \times R^s$, $p + s = \dim N - 1 = n$, and N_1 is diffeomorphic to $T^p \times R^{s+1}$. These yield to the part (4) of the theorem. □

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REZA MIRZAIE
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCES
IMAM KHOMEINI INTERNATIONAL UNIVERSITY (IKIU)
QAZVIN
IRAN

E-mail: mirzaie@ikiu.ac.ir

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