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A general mean value theorem

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Dedicated to the 75th birthday of Professor László Losonczi

Abstract. In this note a general Cauchy-type mean value theorem for the ratio of functional determinants is offered. It generalizes Cauchy's and Taylor's mean value theorems as well as other classical mean value theorems.

1. Introduction

The aim of the present note is to offer a unified approach to most of the mean value theorems known in elementary analysis.

Let $x_1,\dots,x_k$ be arbitrary points in the real interval $[a,b]$. Then, one can determine a permutation π of the set $\{1,\dots,k\}$, $n\in\mathbb{N}$, $\xi_1<\dots<\xi_n$ in $[a,b]$ and $k_1,\dots,k_n$ in $\mathbb{N}$ with $k_1+\dots+k_n=k$ such that

$$(x_{\pi(1)},\dots,x_{\pi(k)})=(\underbrace{\xi_1,\dots,\xi_1}_{k_1\text{ times}},\quad\dots,\quad\underbrace{\xi_n,\dots,\xi_n}_{k_n\text{ times}}). \tag{1}$$

If $w_1,\dots,w_{m+k}:[a,b]\to\mathbb{R}$ is a system of $(k-1)$ times differentiable functions $(m>0)$, and $u_1,\dots,u_{m+k}\in\mathbb{R}^m$ are vectors, then we define

$$\mathcal{W}\begin{pmatrix} w_1, & \dots, & w_{m+k} \\ u_1, & \dots, & u_{m+k}\end{pmatrix}(x_1,\dots,x_k)$$

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$$:= \begin{vmatrix} u_{1,1} & \dots & u_{1,m} & w_1(\xi_1) & \dots & w_1^{(k_1-1)}(\xi_1) & \dots & w_1(\xi_n) & \dots & w_1^{(k_n-1)}(\xi_n) \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ u_{m+k,1} & \dots & u_{m+k,m} & w_{m+k}(\xi_1) & \dots & w_{m+k}^{(k_1-1)}(\xi_1) & \dots & w_{m+k}(\xi_n) & \dots & w_{m+k}^{(k_n-1)}(\xi_n) \end{vmatrix},$$

where the right hand side of this equation is an $(m+k)\times(m+k)$ determinant, $w^{(i)}$ stands for the ith derivative of the function w, $u_{i,j}$ denotes the jth coordinate of the vector u_i, and ξ_i, k_i is determined by (1).

We also allow m to take the value 0, with the following notational conventions: $\mathbb{R}^0 := \{0\}$ and

$$\mathcal{W}\begin{pmatrix} w_1, & \dots, & w_k \\ u_1, & \dots, & u_k \end{pmatrix}(x_1,\dots,x_k) := \mathcal{W}(w_1,\dots,w_k)(x_1,\dots,x_k)$$
$$:= \begin{vmatrix} w_1(\xi_1) & \dots & w_1^{(k_1-1)}(\xi_1) & \dots & w_1(\xi_n) & \dots & w_1^{(k_n-1)}(\xi_n) \\ \vdots & & \vdots & & \vdots & & \vdots \\ w_k(\xi_1) & \dots & w_k^{(k_1-1)}(\xi_1) & \dots & w_k(\xi_n) & \dots & w_k^{(k_n-1)}(\xi_n) \end{vmatrix},$$

Observe that if here $x_1 = \dots = x_k = \xi$, then the above definition reduces to

$$\mathcal{W}(w_1,\dots,w_k)(\xi,\dots,\xi) = \begin{vmatrix} w_1(\xi) & \cdots & w_1^{(k-1)}(\xi) \\ \vdots & \ddots & \vdots \\ w_k(\xi) & \cdots & w_k^{(k-1)}(\xi) \end{vmatrix},$$

which is known as the Wronski determinant of the system $w_1,\dots,w_k$.

The class of functions $f : [a,b] \to \mathbb{R}$ that are $k-1$ times continuously differentiable on $[a,b]$ and k times differentiable on the open interval $]a,b[$ will be denoted by $D^k([a,b])$.

Now we are able to formulate the main result of this paper.

Theorem 1. *Let $1 \le k$, $0 \le m$ be integers and $u_1,\dots,u_{m+k} \in \mathbb{R}^m$ such that (if $0 < m$, then) $u_1,\dots,u_m$ are linearly independent, i.e.,*

$$\mathcal{V}_0 := \begin{vmatrix} u_{1,1} & \cdots & u_{1,m} \\ \vdots & \ddots & \vdots \\ u_{m,1} & \cdots & u_{m,m} \end{vmatrix} \ne 0. \tag{2}$$

In addition, let $w_1,\dots,w_{m+k} \in D^k([a,b])$ be a system of functions satisfying

$$\mathcal{V}_n(\xi) := \mathcal{W}\begin{pmatrix} w_1, & \dots, & w_{m+n} \\ u_1, & \dots, & u_{m+n} \end{pmatrix}(\underbrace{\xi,\dots,\xi}_{n \text{ times}}) \ne 0 \tag{3}$$

for all $\xi \in [a,b]$ *and* $n = 1, \dots, k$. *Then, for all nonidentical points* $x_1, \dots, x_{k+1} \in [a,b]$, *vectors* $p, q \in \mathbb{R}^m$ *and functions* $f, g \in D^k([a,b])$, *there exists an intermediate point* $\xi \in]\min x_i, \max x_i[$ *such that*

$$\begin{aligned}
&\mathcal{W}\begin{pmatrix} w_1, & \dots, & w_{m+k}, & f \\ u_1, & \dots, & u_{m+k}, & p \end{pmatrix}(\xi, \dots, \xi) \\
&\qquad \times \mathcal{W}\begin{pmatrix} w_1, & \dots, & w_{m+k}, & g \\ u_1, & \dots, & u_{m+k}, & q \end{pmatrix}(x_1, \dots, x_{k+1}) \\
&\quad = \mathcal{W}\begin{pmatrix} w_1, & \dots, & w_{m+k}, & g \\ u_1, & \dots, & u_{m+k}, & q \end{pmatrix}(\xi, \dots, \xi) \\
&\qquad \times \mathcal{W}\begin{pmatrix} w_1, & \dots, & w_{m+k}, & f \\ u_1, & \dots, & u_{m+k}, & p \end{pmatrix}(x_1, \dots, x_{k+1}).
\end{aligned} \tag{4}$$

We note that if $m = 0$, then $u_1 = \dots = u_{m+k} = p = q = 0$ by the convention $\mathbb{R}^0 = \{0\}$. Therefore, in this case, the vectors $u_1, \dots, u_{m+k}, p, q$ do not play any role in the formulation of the theorem.

The proof of this theorem is given in the next section. Now we list some of its consequences.

Corollary 1 (Cauchy's Mean Value Theorem)**.** *Let* $f, g \in D^1[a,b]$. *Then there exists* $\xi \in]a,b[$ *such that*

$$f'(\xi)(g(a) - g(b)) = g'(\xi)(f(a) - f(b)).$$

Proof. Let $k = 1$, $m = 0$, $w_1 \equiv 1$ and $x_1 = a$, $x_2 = b$ in Theorem 1. Then the statement follows immediately from (4). □

Corollary 2 (Taylor's Mean Value Theorem)**.** *Let* $f \in D^k([a,b])$. *Then, for all* $x \in]a,b]$, *there exists* $\xi \in]a,b[$ *such that*

$$f(x) = f(a) + f'(a)(x-a) + \dots + \frac{f^{(k-1)}(a)}{(k-1)!}(x-a)^{k-1} + \frac{f^{(k)}(\xi)}{k!}(x-a)^k.$$

Proof. Let $m = 0$,

$$w_i(x) = \frac{(x-a)^{i-1}}{(i-1)!} \quad \text{for} \quad i = 1, \dots, k \quad \text{and} \quad g(x) = \frac{(x-a)^k}{k!}.$$

Then, for all $\xi \in [a,b]$,

$$\mathcal{W}(w_1)(\xi) = \dots = \mathcal{W}(w_1, \dots, w_k)(\xi, \dots, \xi) = 1,$$

therefore, (2) and (3) are satisfied. Thus, taking $x_1 = \cdots = x_k = a$ and $x_{k+1} = x$ in Theorem 1, we obtain that there exists $\xi \in]a, x[$ satisfying

$$\begin{aligned}&\mathcal{W}(w_1, \ldots, w_k, f)(\xi, \ldots, \xi, \xi) \cdot \mathcal{W}(w_1, \ldots, w_k, g)(a, \ldots, a, x) \\ &\quad = \mathcal{W}(w_1, \ldots, w_k, g)(\xi, \ldots, \xi, \xi) \cdot \mathcal{W}(w_1, \ldots, w_k, f)(a, \ldots, a, x).\end{aligned} \tag{5}$$

A simple computation yields that

$$\begin{aligned}&\mathcal{W}(w_1, \ldots, w_k, f)(a, \ldots, a, x) \\ &\qquad = f(x) - f(a) - f'(a)(x-a) - \cdots - \frac{f^{(k-1)}(a)}{(k-1)!}(x-a)^{k-1},\end{aligned}$$

$$\mathcal{W}(w_1, \ldots, w_k, g)(a, \ldots, a, x) = \frac{(x-a)^k}{k!},$$

furthermore

$$\mathcal{W}(w_1, \ldots, w_k, f)(\xi, \ldots, \xi, \xi) = f^{(k)}(\xi) \quad \text{and} \quad \mathcal{W}(w_1, \ldots, w_k, g)(\xi, \ldots, \xi, \xi) = 1.$$

Thus, Taylor's theorem follows from (5) at once. □

Let $w_1(x) = 1, \ldots, w_k(x) = x^{k-1}, w_{k+1} = x^k$ for $x \in [a, b]$ and let $a \le x_1 \le \cdots \le x_{k+1} \le b$ with $a < x_{k+1}$ and $x_1 < b$. Then the ratio

$$[x_1, \ldots, x_{k+1}]f := \frac{\mathcal{W}(w_1, \ldots, w_k, f)(x_1, \ldots, x_k, x_{k+1})}{\mathcal{W}(w_1, \ldots, w_k, w_{k+1})(x_1, \ldots, x_k, x_{k+1})}$$

is called the kth-order divided difference of $f \in D^k([a, b])$ over the points $x_1, \ldots, x_{k+1}$ (c.f. [Sch81, p. 45]). Divided differences are usually defined in an inductive way in the literature, see e.g. [AH79, Sect. 3.17] and [HA38, Sect. 2.3]. The proof of the equivalence of the above definition to the usual one can be found in [Sch81, Theorem 2.51, p. 47].

Concerning divided differences, the following result is well known (c.f. [AH79, p. 274] and [Sch81, (2.93)]).

Corollary 3. *Let $f \in D^k([a, b])$ and $a \le x_1 \le \cdots \le x_{k+1} \le b$ with $x_1 < x_{k+1}$. Then there exists $\xi \in]x_1, x_{k+1}[$ such that*

$$[x_1, \ldots, x_{k+1}]f = \frac{f^{(k)}(\xi)}{k!}. \tag{6}$$

Proof. Apply Theorem 1 when $m = 0$ with the function $g(x) = w_{k+1}(x) = x^k$. Then we find that there exists $\xi \in]x_1, x_{k+1}[$ such that

$$[x_1, \dots, x_{k+1}]f = [\underbrace{\xi, \dots, \xi}_{k+1 \text{ times}}]f = \frac{f^{(k)}(\xi)}{k!}.$$

Thus (6) is proved. □

The following result, called Cauchy Mean Value Theorem, is due to Rätz and Russel [RR87].

Corollary 4. *Let $f, g \in D^k([a,b])$ such that $g^{(k)}(\xi) \neq 0$ for $\xi \in]a,b[$ and let $a \le x_1 \le \dots \le x_{k+1} \le b$ with $x_1 < x_{k+1}$. Then there exists $\xi \in]x_1, x_{k+1}[$ such that*

$$\frac{[x_1, \dots, x_{k+1}]f}{[x_1, \dots, x_{k+1}]g} = \frac{f^{(k)}(\xi)}{g^{(k)}(\xi)}. \tag{7}$$

Proof. Applying Corollary 3 for the function g first, we can observe that

$$[x_1, \dots, x_{k+1}]g \neq 0.$$

Hence the left hand side of (7) exists. Clearly,

$$\frac{[x_1, \dots, x_{k+1}]f}{[x_1, \dots, x_{k+1}]g} = \frac{\mathcal{W}(w_1, \dots, w_k, f)(x_1, \dots, x_k, x_{k+1})}{\mathcal{W}(w_1, \dots, w_k, g)(x_1, \dots, x_k, x_{k+1})}$$

Therefore, by Theorem 1, there exists $\xi \in]x_1, x_{k+1}[$ such that

$$\frac{[x_1, \dots, x_{k+1}]f}{[x_1, \dots, x_{k+1}]g} = \frac{\mathcal{W}(w_1, \dots, w_k, f)(\xi, \dots, \xi, \xi)}{\mathcal{W}(w_1, \dots, w_k, g)(\xi, \dots, \xi, \xi)} = \frac{f^{(k)}(\xi)}{g^{(k)}(\xi)},$$

whence (7) follows. □

2. Proof of the main result

In the proof of Theorem 1, we shall need the following notion: a function $f \in D^k([a,b])$ *vanishes $k+1$ times in* $[a,b]$ if there exist points $x_1 < \dots < x_n$ in $[a,b]$ with $x_1 < b$, $a < x_n$ and natural numbers $k_1, \dots, k_n$ with $k_1 + \dots + k_n = k+1$ such that

$$f^{(j)}(x_i) = 0 \quad \text{for} \quad j = 0, \dots, k_i - 1, \;\; i = 1, \dots, n. \tag{8}$$

For instance, the function $f(x) = x(x-1)$ vanishes twice in $[0,1]$. However, the function $f(x) = x^2$ does not vanish twice in $[0,1]$, but it does in $[-1,1]$, (that is, all the zeroes of f should not be concentrated at the endpoints of the interval).

We recall the following lemmas from [Pal94] and, for readers convenience, we also provide their proofs.

Lemma 1. *If $f, g \in D^k([a,b])$ and f vanishes $k+1$ times in $[a,b]$, then fg also vanishes $k+1$ times in $[a,b]$.*

Proof. By the assumption, there are $x_1 < \cdots < x_n$ in $[a,b]$ with $x_1 < b$, $a < x_n$ and $k_1, \ldots, k_n \in \mathbb{N}$ with $k_1 + \cdots + k_n = k+1$ such that (8) holds. Then, using Leibniz's Product Rule, one can check that

$$(fg)^{(j)}(x_i) = 0 \quad \text{for} \quad j = 0, \ldots, k_i - 1, \;\; i = 1, \ldots, n.$$

Thus fg also vanishes $k+1$ times in $[a,b]$. $\square$

Lemma 2. *If $f \in D^k([a,b])$ vanishes $k+1$ times in $[a,b]$, then f' vanishes k times in $[a,b]$.*

Proof. We have (8) for some $x_1 < \cdots < x_n$ with $x_1 < b$, $a < x_n$ and $k_1, \ldots, k_n \in \mathbb{N}$ with $k_1 + \cdots + k_n = k+1$. If $n = 1$, then there is nothing to prove. Otherwise, by Rolle's Mean Value Theorem, there exist $x_i < \xi_i < x_{i+1}$ such that

$$f'(\xi_i) = 0 \quad \text{for} \quad i = 1, \ldots, n-1.$$

These equalities combined with (8) yield that f' vanishes k times on $[a,b]$. $\square$

The following lemma generalizes [Pal94, Lemma 3]. The result obtained therein corresponds the case $m = 0$ below.

Lemma 3. *Let $1 \le k$, $0 \le m$ be integers and $u_1, \ldots, u_{m+k} \in \mathbb{R}^m$ such that (2) holds (if $m > 0$). Let $w_1, \ldots, w_m \in D^k([a,b])$ be a system of functions satisfying (3) for all $\xi \in [a,b]$. For $f \in D^k([a,b])$, define the following operators*

$$\mathbb{W}_n(f)(\xi) := \mathcal{W}\begin{pmatrix} w_1, & \ldots, & w_{m+n}, & f \\ u_1, & \ldots, & u_{m+n}, & 0 \end{pmatrix}(\underbrace{\xi, \ldots, \xi}_{n+1 \text{ times}}), \qquad n = 0, \ldots, k.$$

where $\xi \in [a,b]$ if $n < k$, and $\xi \in]a,b[$ if $n = k$. Then the following recursive formula

$$\mathbb{W}_n(f)(\xi) = \frac{d}{d\xi}\left(\frac{\mathbb{W}_{n-1}(f)(\xi)}{\mathcal{V}_n(\xi)}\right) \cdot \frac{\left[\mathcal{V}_n(\xi)\right]^2}{\mathcal{V}_{n-1}(\xi)} \tag{9}$$

holds for all $\xi \in [a,b]$ if $1 \le n < k$, and for all $\xi \in]a,b[$ if $n = k$. (Here $\mathcal{V}_0$ and $\mathcal{V}_n$ $(1 \le n \le k)$ are defined in (2) and in (3), respectively. In the case $m = 0$, we set $\mathcal{V}_0 = 0$.)

PROOF. The argument described below works for $m \neq 0$. The case $m = 0$ is completely analogous, therefore it is omitted.

The vectors $u_1, \dots, u_m$ are linearly independent in $\mathbb{R}^m$, hence they form a basis of $\mathbb{R}^m$. Thus, we can find real numbers $\gamma_{1,n}, \dots, \gamma_{m,n}$ such that, for $n = 1, \dots, k$,

$$-u_{m+n} = \gamma_{1,n} u_1 + \dots + \gamma_{m,n} u_m. \tag{10}$$

Then define the functions $v_n : [a, b] \to \mathbb{R}$ by

$$v_n := w_{m+n} + \gamma_{1,n} w_1 + \dots + \gamma_{m,n} w_m. \tag{11}$$

Now we show that the functions $v_1, \dots, v_n$ form a linearly independent system of solutions of the equation

$$\mathbb{W}_n(f)(\xi) = 0, \qquad \xi \in]a, b[\tag{12}$$

which is an nth-order homogeneous linear differential equation for the unknown function f.

To see this, we first compute $\mathbb{W}_n(v_j)$ for any $1 \le j \le k$ and $0 \le n \le k$. Multiplying the first m rows of the determinant $\mathbb{W}_n(v_j)$ by $\gamma_{1,j}, \dots, \gamma_{m,j}$, respectively, subtracting their sum from the last row, then using (10), we get

$$\begin{aligned}
\mathbb{W}_n(v_j)(\xi) &= \mathcal{W}\begin{pmatrix} w_1, & \dots, & w_{m+n}, & w_{m+j} + \sum_{i=1}^m \gamma_{i,j} w_i \\ u_1, & \dots, & u_{m+n}, & 0 \end{pmatrix}(\xi, \dots, \xi) \\
&= \mathcal{W}\begin{pmatrix} w_1, & \dots, & w_{m+n}, & w_{m+j} \\ u_1, & \dots, & u_{m+n}, & -\sum_{i=1}^m \gamma_{i,j} u_i \end{pmatrix}(\xi, \dots, \xi) \\
&= \mathcal{W}\begin{pmatrix} w_1, & \dots, & w_{m+n}, & w_{m+j} \\ u_1, & \dots, & u_{m+n}, & u_{m+j} \end{pmatrix}(\xi, \dots, \xi) = 0
\end{aligned}$$

If $j \le n$, then this formula results $\mathbb{W}_n(v_j) = 0$. On the other hand, with $j = n+1$, we have $\mathbb{W}_n(v_{n+1}) = \mathcal{V}_{n+1}$.

The function v_1 cannot be identically zero because $\mathbb{W}_0(v_1) = \mathcal{V}_1 \neq 0$. Hence $\{v_1\}$ is a linearly independent set of solutions of $\mathbb{W}_1(f) = 0$. Assume now that $v_1, \dots, v_n$ form a linearly independent system of solutions of $\mathbb{W}_n(f) = 0$. The function v_{n+1} is not a solution of this equation, hence, it cannot be a linear combination of $v_1, \dots, v_n$. Thus, $v_1, \dots, v_{n+1}$ is also a linearly independent system.

Temporarily, denote the operator defined by the right hand side of (9) by $\mathbb{W}_n^*(f)$. It is clear that $\mathbb{W}_n^*(f)$ is also an nth-order linear differential operator of f. We show that $v_1, \dots, v_n$ also solves the equation $\mathbb{W}_n^*(f) = 0$. This is obvious if $f = v_1, \dots, v_{n-1}$ (since these functions are solutions of the equation $\mathbb{W}_{n-1}(f) = 0$). On the other hand,

$$\mathbb{W}_n^*(v_n)(\xi) = \frac{d}{d\xi}\left(\frac{\mathbb{W}_{n-1}(v_n)(\xi)}{\mathcal{V}_n(\xi)}\right) \cdot \frac{\left[\mathcal{V}_n(\xi)\right]^2}{\mathcal{V}_{n-1}(\xi)} = \frac{d}{d\xi}\left(\frac{\mathcal{V}_n(\xi)}{\mathcal{V}_n(\xi)}\right) \cdot \frac{\left[\mathcal{V}_n(\xi)\right]^2}{\mathcal{V}_{n-1}(\xi)} = 0.$$

Observe that the coefficients of $f^{(n)}$ in $\mathbb{W}_n(f)$ and $\mathbb{W}_n^*(f)$ are equal to $\mathcal{V}_n$ which does not vanish anywhere in $[a, b]$. Therefore, having the same solution space, these two operators have to coincide for all $1 \le n \le k$. $\square$

Lemma 4. *Let $1 \le k$, $0 \le m$ be integers and $u_1, \dots, u_{m+k} \in \mathbb{R}^m$ such that* (2) *holds (if $m > 0$). Let $w_1, \dots, w_m \in D^k([a, b])$ be a system of functions satisfying* (3) *for all $\xi \in [a, b]$. Assume that the function $f \in D^k([a, b])$ vanishes $k + 1$ times in $[a, b]$. Then, for each $0 \le n \le k$, the function $\mathbb{W}_n(f)$ defined in Lemma 3 vanishes $k + 1 - n$ times in $[a, b]$.*

Proof. We prove by induction. If $n = 0$, then $\mathbb{W}_0(f) = \mathcal{V}_0 f$, hence, in this case, there is nothing to prove. Let $n \ge 1$ and assume that $\mathbb{W}_{n-1}(f)$ vanishes $k + 1 - (n - 1)$ times. Then, applying Lemma 1 and Lemma 2, one sees that the function

$$\frac{d}{d\xi}\left(\frac{\mathbb{W}_{n-1}(f)(\xi)}{\mathcal{V}_n(\xi)}\right) \cdot \frac{\left[\mathcal{V}_n(\xi)\right]^2}{\mathcal{V}_{n-1}(\xi)} \qquad (\xi \in [a, b])$$

vanishes $k + 1 - (n - 1) - 1 = k + 1 - n$ times. Therefore, due to the recursive formula established in Lemma 3, $\mathbb{W}_n(f)$ vanishes $k + 1 - n$ times. $\square$

Now we are ready to prove the main result of the paper.

Proof of Theorem 1. Let $x_1 \le \dots \le x_{k+1}$ be in $[a, b]$ with $\min x_i < \max x_i$. Then there exist a permutation π of $\{1, \dots, x_{k+1}\}$, $n \in \mathbb{N}$, $\xi_1 < \dots < \xi_n$ in $[a, b]$ and $k_1, \dots, k_n \in \mathbb{N}$ with $k_1 + \dots + k_n = k + 1$ such that

$$(x_{\pi(1)}, \dots, x_{\pi(k+1)}) = (\underbrace{\xi_1, \dots, \xi_1}_{k_1 \text{ times}}, \quad \dots, \quad \underbrace{\xi_n, \dots, \xi_n}_{k_n \text{ times}}) \tag{13}$$

holds. Define the function $F : [a, b] \to \mathbb{R}$ by

$$F(x)$$

$$:= \begin{vmatrix} u_{1,1} & \dots & u_{1,m} & w_1(\xi_1) & \dots & w_1^{(k_1-1)}(\xi_1) & \dots & w_1^{(k_n-1)}(\xi_n) & w_1(x) \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\ u_{m+k,1} & \dots & u_{m+k,m} & w_{m+k}(\xi_1) & \dots & w_{m+k}^{(k_1-1)}(\xi_1) & \dots & w_{m+k}^{(k_n-1)}(\xi_n) & w_{m+k}(x) \\ p_1 & \dots & p_m & f(\xi_1) & \dots & f^{(k_1-1)}(\xi_1) & \dots & f^{(k_n-1)}(\xi_n) & f(x) \\ q_1 & \dots & q_m & g(\xi_1) & \dots & g^{(k_1-1)}(\xi_1) & \dots & g^{(k_n-1)}(\xi_n) & g(x) \end{vmatrix}.$$

It is obvious at once that

$$F^{(j)}(\xi_i) = 0 \quad \text{for} \quad j = 0, \dots, k_i - 1, \;\; i = 1, \dots, n,$$

therefore F vanishes $k+1$ times in $[a, b]$. Thus, by Lemma 4, there exists $\xi \in]a, b[$ such that

$$\mathbb{W}_k(F)(\xi) = \mathcal{W}\begin{pmatrix} w_1, & \dots, & w_{m+k}, & F \\ u_1, & \dots, & u_{m+k}, & 0 \end{pmatrix}(\underbrace{\xi, \dots, \xi}_{k+1 \text{ times}}) = 0. \tag{14}$$

Now determine the contstants $\gamma_{i,n}$ such that they satisfy (10) and define $v_1, \dots, v_k$ by (11). Similarly, choose $\alpha_1, \dots, \alpha_m$ and $\beta_1, \dots, \beta_m$ such that

$$-p = \alpha_1 u_1 + \dots + \alpha_m u_m \quad \text{and} \quad -q = \beta_1 u_1 + \dots + \beta_m u_m. \tag{15}$$

and define

$$\phi = f + \alpha_1 w_1 + \dots + \alpha_m w_m \quad \text{and} \quad \psi = g + \beta_1 w_1 + \dots + \beta_m w_m. \tag{16}$$

Using these choices of the constants, add linear combination of the first m rows of F to the rest of the rows to obtain

$$F(x)$$

$$:= \begin{vmatrix} u_{1,1} & \dots & u_{1,m} & w_1(\xi_1) & \dots & w_1^{(k_1-1)}(\xi_1) & \dots & w_1^{(k_n-1)}(\xi_n) & w_1(x) \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\ u_{m,1} & \dots & u_{m,m} & w_m(\xi_1) & \dots & w_m^{(k_1-1)}(\xi_1) & \dots & w_m^{(k_n-1)}(\xi_n) & w_m(x) \\ 0 & \dots & 0 & v_1(\xi_1) & \dots & v_1^{(k_1-1)}(\xi_1) & \dots & v_1^{(k_n-1)}(\xi_n) & v_1(x) \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & v_k(\xi_1) & \dots & v_k^{(k_1-1)}(\xi_1) & \dots & v_k^{(k_n-1)}(\xi_n) & v_k(x) \\ 0 & \dots & 0 & \phi(\xi_1) & \dots & \phi^{(k_1-1)}(\xi_1) & \dots & \phi^{(k_n-1)}(\xi_n) & \phi(x) \\ 0 & \dots & 0 & \psi(\xi_1) & \dots & \psi^{(k_1-1)}(\xi_1) & \dots & \psi^{(k_n-1)}(\xi_n) & \psi(x) \end{vmatrix}$$

$$= \mathcal{V}_0 \cdot \begin{vmatrix} v_1(\xi_1) & \dots & v_1^{(k_1-1)}(\xi_1) & \dots & v_1^{(k_n-1)}(\xi_n) & v_1(x) \\ \vdots & & \vdots & & \vdots & \vdots \\ v_k(\xi_1) & \dots & v_k^{(k_1-1)}(\xi_1) & \dots & v_k^{(k_n-1)}(\xi_n) & v_k(x) \\ \phi(\xi_1) & \dots & \phi^{(k_1-1)}(\xi_1) & \dots & \phi^{(k_n-1)}(\xi_n) & \phi(x) \\ \psi(\xi_1) & \dots & \psi^{(k_1-1)}(\xi_1) & \dots & \psi^{(k_n-1)}(\xi_n) & \psi(x) \end{vmatrix}.$$

Expanding by the last column, we get

$$F(x) = \sum_{i=1}^{k} C_i \cdot v_i(x) - A\phi(x) + B\psi(x),$$

where A, B, C_i are the values of the corresponding subdeterminants. Substituting the above form of F into (14), and using that $\mathbb{W}_k(v_i) = 0$, we get that

$$A \cdot \mathbb{W}_k(\phi)(\xi) = B \cdot \mathbb{W}_k(\psi)(\xi) \tag{17}$$

In the rest of the proof we show that (17) reduces to (4).

Indeed, adding a certain linear combination to the last row of $\mathbb{W}_k$, we get

$$\begin{aligned}
\mathbb{W}_k(\phi)(\xi) &= \mathcal{W}\begin{pmatrix} w_1, & \dots, & w_{m+k}, & f + \sum_{i=1}^{m} \alpha_i w_i \\ u_1, & \dots, & u_{m+k}, & 0 \end{pmatrix}(\xi, \dots, \xi) \\
&= \mathcal{W}\begin{pmatrix} w_1, & \dots, & w_{m+k}, & f \\ u_1, & \dots, & u_{m+k}, & -\sum_{i=1}^{m} \alpha_i u_i \end{pmatrix}(\xi, \dots, \xi) \\
&= \mathcal{W}\begin{pmatrix} w_1, & \dots, & w_{m+k}, & f \\ u_1, & \dots, & u_{m+k}, & p \end{pmatrix}(\xi, \dots, \xi).
\end{aligned}$$

For the constant A, due to its origin, we have

$$A = \mathcal{V}_0 \cdot \begin{vmatrix} v_1(\xi_1) & \dots & v_1^{(k_1-1)}(\xi_1) & \dots & v_1^{(k_n-1)}(\xi_n) \\ \vdots & & \vdots & & \vdots \\ v_k(\xi_1) & \dots & v_k^{(k_1-1)}(\xi_1) & \dots & v_k^{(k_n-1)}(\xi_n) \\ \psi(\xi_1) & \dots & \psi^{(k_1-1)}(\xi_1) & \dots & \psi^{(k_n-1)}(\xi_n) \end{vmatrix}.$$

Now, using an argument similar to that applied in the computation of F, one can get that

$$A = \begin{vmatrix} u_{1,1} & \dots & u_{1,m} & w_1(\xi_1) & \dots & w_1^{(k_1-1)}(\xi_1) & \dots & w_1^{(k_n-1)}(\xi_n) \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ u_{m,1} & \dots & u_{m,m} & w_m(\xi_1) & \dots & w_m^{(k_1-1)}(\xi_1) & \dots & w_m^{(k_n-1)}(\xi_n) \\ 0 & \dots & 0 & v_1(\xi_1) & \dots & v_1^{(k_1-1)}(\xi_1) & \dots & v_1^{(k_n-1)}(\xi_n) \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & v_k(\xi_1) & \dots & v_k^{(k_1-1)}(\xi_1) & \dots & v_k^{(k_n-1)}(\xi_n) \\ 0 & \dots & 0 & \psi(\xi_1) & \dots & \psi^{(k_1-1)}(\xi_1) & \dots & \psi^{(k_n-1)}(\xi_n) \end{vmatrix}$$

$$= \begin{vmatrix} u_{1,1} & \dots & u_{1,m} & w_1(\xi_1) & \dots & w_1^{(k_1-1)}(\xi_1) & \dots & w_1^{(k_n-1)}(\xi_n) \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ u_{m+k,1} & \dots & u_{m+k,m} & w_{m+k}(\xi_1) & \dots & w_{m+k}^{(k_1-1)}(\xi_1) & \dots & w_{m+k}^{(k_n-1)}(\xi_n) \\ q_1 & \dots & q_m & g(\xi_1) & \dots & g^{(k_1-1)}(\xi_1) & \dots & g^{(k_n-1)}(\xi_n) \end{vmatrix}.$$

$$= \mathcal{W}\begin{pmatrix} w_1, & \dots, & w_{m+k}, & g \\ u_1, & \dots, & u_{m+k}, & q \end{pmatrix}(x_1, \dots, x_{k+1}).$$

Thus, we have checked that the left hand side of (17) coincides with that of (4). The equality of the right hand sides follows similarly, and therefore, the proof is complete. □

We now derive a useful consequence of Theorem 1.

Theorem 2. *Let $I \subset \mathbb{R}$ be an interval and $[a, b]$ be a proper subinterval of I. Let $1 \le k$, $1 \le m$ be integers and $y_1, \dots, y_m \in I \setminus [a, b]$. Assume that $w_1, \dots, w_{m+k} : I \to \mathbb{R}$ are sufficently many times differentiable functions such that*

$$\mathcal{W}(w_1, \dots, w_{m+n})(y_1, \dots, y_m, \underbrace{\xi, \dots, \xi}_{n \text{ times}}) \neq 0 \tag{18}$$

for all $\xi \in [a, b]$ and $n = 0, \dots, k$. Then, for all nonidentical points $x_1, \dots, x_{k+1} \in [a, b]$, and functions sufficiently many times differentiable $f, g : I \to \mathbb{R}$, there exists an intermediate point $\xi \in]\min x_i, \max x_i[$ such that

$$\begin{aligned} &\mathcal{W}(w_1, \dots, w_{m+k}, f)(y_1, \dots, y_m, \underbrace{\xi, \dots, \xi}_{k+1 \text{ times}}) \\ &\qquad \times \mathcal{W}(w_1, \dots, w_{m+k}, g)(y_1, \dots, y_m, x_1, \dots, x_{k+1}) \\ &\quad = \mathcal{W}(w_1, \dots, w_{m+k}, g)(y_1, \dots, y_m, \underbrace{\xi, \dots, \xi}_{k+1 \text{ times}}) \\ &\qquad \times \mathcal{W}(w_1, \dots, w_{m+k}, f)(y_1, \dots, y_m, x_1, \dots, x_{k+1}). \end{aligned} \tag{19}$$

Proof. Let π be a permutation of the set $\{1, \dots, m\}$, $\eta_1, \dots, \eta_l \in I$, and $m_1, \dots, m_l \in \mathbb{N}$ with $m_1 + \dots + m_l = m$ such that

$$(y_{\pi(1)}, \dots, y_{\pi(m)}) = (\underbrace{\eta_1, \dots, \eta_1}_{m_1 \text{ times}}, \quad \dots, \quad \underbrace{\eta_l, \dots, \eta_l}_{m_l \text{ times}}).$$

Set, for $i = 1, \dots, m + k$,

$$u_i := (u_{i1}, \dots, u_{im})$$

$$:= (w_i(\eta_1), \dots, w_i^{(m_1-1)}(\eta_1), \quad \dots, \quad w_i(\eta_l), \dots, w_i^{(m_l-1)}(\eta_l)),$$

and

$$\begin{aligned} p &:= (p_1, \dots, p_m) \\ &:= (f(\eta_1), \dots, f^{(m_1-1)}(\eta_1), \quad \dots, \quad f(\eta_l), \dots, f^{(m_l-1)}(\eta_l)), \\ q &:= (q_1, \dots, q_m) \\ &:= (g(\eta_1), \dots, g^{(m_1-1)}(\eta_1), \quad \dots, \quad g(\eta_l), \dots, g^{(m_l-1)}(\eta_l)). \end{aligned}$$

Observe, that with this notations, the conditions of Theorem 1 are satisfied and therefore there exists ξ such that (4) holds. It is immediate to see that (4) is equivalent to (19). $\square$

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