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Nonoscillation theorems for second order quasilinear differential equations

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Abstract. Some nonoscillation criteria are obtained for second order quasilinear differential equations of the form

$$[r(t)|u'(t)|^{p-2}u'(t)]' + c(t)|u(t)|^{p-2}u(t) = 0, \tag{E}$$

where $p > 1$, $r(t) \in C^1([t_0,\infty);(0,\infty))$ and $c(t) \in C([t_0,\infty);\mathbb{R})$. These results extend some nonoscillation criteria of Hille, Wintner, Potter, Moore, Willett for the equation

$$[r(t)u'(t)]' + c(t)u(t) = 0$$

to equation (E).

1. Introduction

In this paper, we consider the following second order quasilinear differential equation

$$[r(t)|u'(t)|^{p-2}u'(t)]' + c(t)|u(t)|^{p-2}u(t) = 0, \tag{E}$$

where $p > 1$ is a constant, $r(t)\in C^1([t_0,\infty);(0,\infty))$ and $c(t)\in C([t_0,\infty);\mathbb{R})$ for some $t_0 \geq 0$. If $p = 2$, then equation (E) reduces to the linear differential equation

$$[r(t)u'(t)]' + c(t)u(t) = 0. \tag{E$_1$}$$

A solution of (E) is a function $u \in C^1([\,t_0,\infty),\mathbb{R})$ with $r|u'|^{p-2}u' \in C^1[t_0,\infty)$ and satisfies equation (E) on $[t_0,\infty)$. In [1], Elbert established

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the existence and uniqueness of solutions to the initial value problem for (E) on $[t_0, \infty)$. It follows from (E) that any constant multiple of a solution of (E) is also a solution. A solution u(t) of (E) is said to be nonoscillatory if there is a number $T \geq t_0$ such that $u(t) \neq 0$ for $t \geq T$. Equation (E) is said to be nonoscillatory if all its solutions are nonoscillatory.

There is a striking similarity in the oscillatory behavior between the second order quasilinear differential equation (E) and the corresponding linear equation (E_1), see, for example, Elbert [1,2], Mirzov [9,10] and Li and Yeh [7]. For example, Sturmian comparison and separation theorems for (E_1), see for example [13], have been extended in a natural way to (E). Thus all solutions of (E) are either oscillatory or nonoscillatory, that is, the consistency of oscillatory and nonoscillatory solutions is excluded for equation (E).

For more recent papers of such similarity between (E) and (E_1), we refer to Kusano et al [4, 5, 6], Li and Yeh [8].

In [7], the present authors established the following sufficient and necessary condition on the nonoscillation of (E).

Theorem 1.1 ([7], Theorem 3.2). *Equation* (E) *is nonoscillatory if and only if there are a number $T \geq t_0$ and a function $f \in C^1[T, \infty)$ satisfying*

$$c(t) + (p-1)r(t)|f(t)|^q - [r(t)f(t)]' \leq 0 \qquad \text{for } t \geq T,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

The purpose of this paper is to establish some nonoscillation criteria of (E) by using Theorem 1.1. These results improve some nonoscillation criteria of Hille [3], Wintner [15], Potter [12], Moore [11], and Willett [14], of (E_1) to equation (E).

2. Nonoscillation criteria

We assume, throughout this paper, that $\frac{1}{p} + \frac{1}{q} = 1$ and $\varepsilon = (q-1)q^{-p}$.

Theorem 2.1. *Let $g(t)$ and $\psi(t)$ be two continuously differentiable functions on $[t_0, \infty)$ satisfying $g(t) > 0$, $g'(t) \geq r^{1-q}(t)$ and $\psi'(t) \leq -c(t)$. If*

$$\limsup_{t \to \infty} \; g^{p-1}(t)|\psi(t)| < \varepsilon, \tag{2.1}$$

then equation (E) *is nonoscillatory.*

Proof. By (2.1), there are two numbers $T \geq t_0$ and $k \in (0,\varepsilon)$ such that $|\psi(t)| < kg^{1-p}(t)$ for $t \geq T$. Let

$$f = -\frac{1}{\lambda r}\left(\lambda\psi + \frac{1-\lambda k}{g^{p-1}}\right),$$

where $\lambda = q^{p-1}$. Then

$$\lambda k < \lambda\varepsilon = q^{p-1}(q-1)q^{-p} = \frac{q-1}{q} < 1, \tag{1}$$

$$(rf)' = -\psi' + \frac{(p-1)(1-\lambda k)g'}{\lambda g^p} \geq c + \frac{(p-1)(1-\lambda k)}{\lambda r^{q-1}g^p}$$

and

$$\begin{aligned}
c + (p-1)r|f|^q - (rf)' &\leq c + (p-1)r\left|\frac{1}{\lambda r}\left(\lambda\psi + \frac{1-\lambda k}{g^{p-1}}\right)\right|^q \\
-c - \frac{(p-1)(1-\lambda k)}{\lambda r^{q-1}g^p} &= (p-1)r^{1-q}g^{-p}\left\{\left|\psi g^{p-1} + \frac{1-\lambda k}{\lambda}\right|^q - \frac{1-\lambda k}{\lambda}\right\} \\
&\leq (p-1)r^{1-q}g^{-p}\left\{\left|k + \frac{1-\lambda k}{\lambda}\right|^q - \frac{1-\lambda k}{\lambda}\right\} \\
&= (p-1)r^{1-q}g^{-p}\left(\frac{1}{\lambda^q} - \frac{1}{\lambda} + k\right) = (p-1)r^{1-q}g^{-p}(k-\varepsilon) \leq 0.
\end{aligned}$$

It follows from Theorem 1.1 that equation (E) is nonoscillatory.

If $p = 2$, $r(t) = 1$, $c(t) > 0$ for $t \geq t_0$ and $g(t) = t$, $\psi(t) = \int_t^\infty c(s)ds < \infty$, then Theorem 2.1 reduces to a Hille's criterion [3, p. 246].

Theorem 2.2. *Let $g(t)$ and $\psi(t)$ be two continuously differentiable functions on $[t_0,\infty)$ satisfying $g(t) > 0$, $g'(t) \leq -r^{1-q}(t)$ and $\psi'(t) \geq c(t)$. If* (2.1) *holds, then equation* (E) *is nonoscillatory.*

Proof is the same as in Theorem 2.1 with the exception that $f = \frac{1}{\lambda r}\left(\lambda\psi + \frac{1-\lambda k}{g^{p-1}}\right)$ and $\lambda k \leq \lambda\varepsilon$ in (1).

Theorem 2.3. *Let $g(t)$ and $\psi(t)$ be two continuously differentiable functions on $[t_0,\infty)$ satisfying $g(t) > 0$, $g'(t) \geq r^{1-q}(t)$ and $\psi'(t) \leq -c(t)$. If there exists a number $k > 0$ such that*

$$-k^{\frac{1}{q}} - k \leq g^{p-1}(t)\psi(t) \leq k^{\frac{1}{q}} - k \leq \varepsilon. \tag{2.2}$$

then equation (E) *is nonoscillatory.*

Proof. Let

$$f = -\frac{1}{r}\left(\psi + \frac{k}{g^{p-1}}\right).$$

Then

$$(rf)' = -\psi' + \frac{(p-1)kg'}{g^p} \geq c + \frac{(p-1)k}{r^{q-1}g^p}$$

and

$$c + (p-1)r|f|^q - (rf)' \leq c + (p-1)r\left|\frac{1}{r}\left(\psi + \frac{k}{g^{p-1}}\right)\right|^q - c - \frac{(p-1)k}{r^{q-1}g^p}$$

$$= (p-1)r^{1-q}g^{-p}\left\{\left|\psi g^{p-1} + k\right|^q - k\right\} \leq (p-1)r^{1-q}g^{-p}(k-k) = 0.$$

It follows from Theorem 1.1 that equation (E) is nonoscillatory.

If $p = 2,\ g(t) = 1 + \int_{t_0}^{t} \frac{1}{r(s)} ds$ as $t \to \infty$ and $\psi(t) = \int_t^\infty c(s)ds < \infty$, then Theorem 2.3 reduces to a Moore's criterion [11, Theorem 6].

Theorem 2.4. *Let $g(t)$ and $\psi(t)$ be two continuously differentiable functions on $[t_0, \infty)$ satisfying $g(t) > 0$, $g'(t) \leq -r^{1-q}(t)$ and $\psi'(t) \geq c(t)$. If there exists a number $k > 0$ such that* (2.2) *holds, then equation* (E) *is nonoscillatory.*

Proof. Let

$$f = \frac{1}{r}\left(\psi + \frac{k}{g^{p-1}}\right).$$

Then

$$(rf)' = \psi' - \frac{(p-1)kg'}{g^p} \geq c + \frac{(p-1)k}{r^{q-1}g^p}$$

and

$$c + (p-1)r|f|^q - (rf)' \leq c + (p-1)r\left|\frac{1}{r}\left(\psi + \frac{k}{g^{p-1}}\right)\right|^q - c - \frac{(p-1)k}{r^{q-1}g^p}$$

$$= (p-1)r^{1-q}g^{-p}\left\{\left|\psi g^{p-1} + k\right|^q - k\right\} \leq (p-1)r^{1-q}g^{-p}(k-k) = 0.$$

It follows from Theorem 1.1 that equation (E) is nonoscillatory.

If $p = 2,\ g(t) = \int_t^\infty \frac{1}{r(s)} ds < \infty$ and $\psi(t) = \int_{t_0}^{t} c(s)ds$, then Theorem 2.4 reduces to a Moore's criterion [11, Theorem 6].

Example 2.5. Consider the equation

$$(\mathrm{E}_2) \qquad \left(t^\alpha|u'|^{p-2}u'\right)' + \lambda t^{\alpha-p}|u|^{p-2}u = 0, \quad \text{for } t > 0$$

where $\alpha > p-1$ and $\lambda > 0$ are two constants. Let

$$g(t) = \int_t^\infty s^{-\frac{\alpha}{p-1}}\,ds$$

and

$$\psi(t) = \int_0^t \lambda s^{\alpha-p} ds.$$

Then

$$g(t) = \frac{p-1}{\alpha-p+1} t^{\frac{p-1-\alpha}{p-1}}, \qquad \psi(t) = \frac{\lambda}{\alpha-p+1} t^{\alpha-p+1}$$

and

$$g^{p-1}(t)\psi(t) = \frac{\lambda}{\alpha-p+1}\left(\frac{p-1}{\alpha-p+1}\right)^{p-1}.$$

If

$$\lambda \le \left(\frac{\alpha-p+1}{p}\right)^p,$$

then $g^{p-1}(t)\psi(t) \le \varepsilon$. It follows from Theorem 2.4 that equation (E_2) is nonoscillatory.

Theorem 2.6. *Let $g(t)$ be a continuously differentiable function on $[t_0,\infty)$ such that $g(t) > 0$ and $g'(t) \ge r^{1-q}(t)$. If there exists a continuously differentiable function $\psi(t)$ on $[t_0,\infty)$ such that $\lim_{t\to\infty}\psi(t)$ exists and $\psi'(t) \le -g^{p-1}(t)c(t)$, then equation* (E) *is nonoscillatory.*

Proof. Since $\lim_{t\to\infty}\psi(t)$ exists, there exist two real numbers $T \ge t_0$ and M such that $0 < M + \psi(t) \le 1$ for $t \ge T$. Let

$$f = -\frac{M+\psi}{rg^{p-1}}.$$

Then

$$(rf)' = \frac{(p-1)(M+\psi)g'}{g^p} - \frac{\psi'}{g^{p-1}} \ge c + \frac{(p-1)(M+\psi)}{r^{q-1}g^p}$$

which implies

$$c+(p-1)r|f|^q-(rf)' \le c+(p-1)r\left|\frac{M+\psi}{rg^{p-1}}\right|^q - c - \frac{(p-1)(M+\psi)}{r^{q-1}g^p}$$
$$\le (p-1)r^{1-q}g^{-p}[(M+\psi)-(M+\psi)] = 0.$$

It follows from Theorem 1.1 that equation (E) is nonoscillatory.

Theorem 2.7. *Let $g(t)$ be a continuously differentiable function on $[t_0,\infty)$ such that $g(t)>0$ and $g'(t)\le -r^{1-q}(t)$. If there exists a continuously differentiable function $\psi(t)$ such that $\lim_{t\to\infty}\psi(t)$ exists and $\psi'(t)\le -g^{p-1}(t)c(t)$, then equation* (E) *is nonoscillatory.*

Proof. Since $\lim_{t\to\infty}\psi(t)$ exists, there exist two real numbers $T\ge t_0$ and M such that $0<M-\psi(t)\le 1$ for $t\ge T$. Let

$$f=\frac{M-\psi}{rg^{p-1}}.$$

Then

$$(rf)' = -\frac{(p-1)(M-\psi)g'}{g^p} - \frac{\psi'}{g^{p-1}} \ge c + \frac{(p-1)(M-\psi)}{r^{q-1}g^p},$$

which implies

$$c+(p-1)r|f|^q-(rf)' \le c+(p-1)r\left|\frac{M-\psi}{rg^{p-1}}\right|^q - c - \frac{(p-1)(M-\psi)}{r^{q-1}g^p}$$
$$\le (p-1)r^{1-q}g^{-p}[(M-\psi)-(M-\psi)] = 0.$$

It follows from Theorem 1.1 that equation (E) is nonoscillatory.

Theorem 2.8. *Let*

$$c(t)\le \frac{1}{h^p(t)},$$

where $h(t)\in C^1([t_0,\infty);(0,\infty))$. If either

$$h'(t)-\frac{1}{r^{q-1}(t)} \ge \frac{1}{p-1} \qquad \text{for all } t \text{ large enough,}$$

or

$$\lim_{t\to\infty}\left(h'(t)-\frac{1}{r^{q-1}(t)}\right) = L \text{ exists} \quad \text{and} \quad L>\frac{1}{p-1},$$

then equation (E) *is nonoscillatory.*

PROOF. It follows from the assumption that there is a number $T \geq t_0$ such that

$$h'(t) - \frac{1}{r^{q-1}(t)} \geq \frac{1}{p-1} \qquad \text{for all } t \geq T.$$

Let

$$f = -\frac{1}{rh^{p-1}}.$$

Then

$$\begin{aligned} c + (p-1)r|f|^q - (rf)' &\leq \frac{1}{h^p} + \frac{p-1}{r^{q-1}h^p} - \frac{(p-1)h'}{h^p} \\ &= (p-1)h^{-p}\left(\frac{1}{p-1} + \frac{1}{r^{q-1}} - h'\right) \leq 0 \quad \text{for } t \geq T. \end{aligned}$$

Hence, by Theorem 1.1, equation (E) is nonoscillatory.

If $p = 2$ and $r(t) = 1$, then Theorem 2.7 reduces to a POTTER's criterion [12, Theorem 1.5].

Theorem 2.9. *Let $\psi(t)$ be a nonnegative continuously differentiable function on $[t_0, \infty)$ such that $\psi'(t) \leq -c(t)$. If*

$$\int_t^\infty \frac{\psi^q(s)}{r^{q-1}(s)} ds \leq p^{-q}\psi(t),$$

then equation (E) *is nonoscillatory.*

PROOF. Let

$$f = -\frac{1}{r(t)}\left(\psi(t) + (p-1)p^q \int_t^\infty \frac{\psi^q(s)}{r^{q-1}(s)} ds\right).$$

Then

$$[r(t)f(t)]' \geq c(t) + \frac{(p-1)p^q\psi^q(t)}{r^{q-1}(t)}.$$

Hence

$$
\begin{aligned}
&c(t)+(p-1)r(t)|f(t)|^q-[r(t)f(t)]' \\
&\quad \le c(t)+(p-1)r(t)\left|\frac{1}{r(t)}\left(\psi(t)+(p-1)p^q\int_t^\infty \frac{\psi^q(s)}{r^{q-1}(s)}ds\right)\right|^q \\
&\qquad - c(t)-\frac{(p-1)p^q\psi^q(t)}{r^{q-1}(t)} \\
&\quad =(p-1)r^{1-q}(t)\left\{\left|\psi(t)+(p-1)p^q\int_t^\infty \frac{\psi^q(s)}{r^{q-1}(s)}ds\right|^q-p^q\psi^q(t)\right\} \\
&\quad \le (p-1)r^{1-q}(t)\Big[|\psi(t)+(p-1)p^q\cdot p^{-q}\psi(t)|^q-p^q\psi^q(t)\Big]=0.
\end{aligned}
$$

It follows from Theorem 1.1 that equation (E) is nonoscillatory.

Theorem 2.10. *Let $\psi(t)$ be a nonnegative continuously differentiable function on $[t_0,\infty)$ such that $\psi'(t)\le -c(t)$, and let*

$$
\psi_1(t)=\int_t^\infty \frac{\psi^q(s)}{r^{q-1}(s)}\exp\left((p-1)p^{q-1}\int_t^s \frac{\psi^{q-1}(\xi)}{r^{q-1}(\xi)}d\xi\right)ds.
$$

If $\psi_1(t)\le p^{1-q}\psi(t)$, then equation (E) *is nonoscillatory.*

Proof. Let

$$
f=-\frac{1}{r}\left(\psi+(p-1)p^{q-1}\psi_1\right).
$$

Then

$$
\begin{aligned}
(rf)'&=-\psi'-(p-1)p^{q-1}\psi_1' \\
&\ge c+(p-1)p^{q-1}r^{1-q}\left((p-1)p^{q-1}\psi^{q-1}\psi_1+\psi^q\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&c+(p-1)r|f|^q-(rf)'\le\ c+(p-1)r^{1-q}[\psi+(p-1)p^{q-1}\psi_1]^q \\
&\qquad -c-(p-1)p^{q-1}r^{1-q}[(p-1)p^{q-1}\psi^{q-1}\psi_1+\psi^q] \\
&\quad =\ (p-1)r^{1-q}[\psi+(p-1)p^{q-1}\psi_1]\big\{[\psi+(p-1)p^{q-1}\psi_1]^{q-1}-p^{q-1}\psi^{q-1}\big\} \\
&\quad \le\ (p-1)r^{1-q}[\psi+(p-1)p^{q-1}\psi_1]\big\{[\psi+(p-1)p^{q-1}\cdot p^{1-q}\psi]^{q-1} \\
&\qquad -p^{q-1}\psi^{q-1}\big\} \\
&\quad =\ (p-1)r^{1-q}[\psi+(p-1)p^{q-1}\psi_1][p^{q-1}\psi^{q-1}-p^{q-1}\psi^{q-1}]=0.
\end{aligned}
$$

It follows from Theorem 1.1 that equation (E) is nonoscillatory.

References

[1] Á. ELBERT, A half-linear second order differential equation, *Colloquia Math. Soc. János Bolyai 30:* Qualitive Theory of Differential Equations, *Szeged*, (1979), 153–180.

[2] Á. ELBERT, Oscillation and nonoscillation theorems for some nonlinear ordinary differential equations, *Lecture Notes in Mathematics,* **964:** Ordinary and Partial Differential Equations (1982), 187–212.

[3] E. HILLE, Non-oscillation theorems, *Trans. Amer. Math. Soc.* **64** (1948), 234–252.

[4] T. KUSANO, Y. NAITO and A. OGATA, Strong oscillation and nonoscillation of quasilinear differential equation of second order, *Differential Equations and Dynamical Systems* **2** (1994), 1–10.

[5] T. KUSANO and Y. NAITO, Oscillation and nonoscillation criteria for second order quasilinear differential equations, (*preprint*).

[6] T. KUSANO and N. YOSHIDA, Nonoscillation theorems for a class of quasilinear differential equations of second order, *J. Math. Anal. Appl.* **189** (1995), 115–127.

[7] H. J. LI and C. C. YEH, Sturmian comparison theorem for half-linear second order differential equations, *Proc. of the Royal Soc. of Edinburgh* (*to appear*).

[8] H. J. LI and C. C. YEH, An integral criterion for oscillation of nonlinear differential equations, *Japonica Math.* **41** (1995), 185–188.

[9] D. D. MIRZOV, On some analogs of Sturm's and Kneser's theorems for nonlinear systems, *J. Math. Anal. Appl.* **53** (1976), 418–425.

[10] D. D. MIRZOV, On the oscillation of solutions of a system of differential equations, *Mat. Zametki* **23** (1978), 401–404.

[11] P. A. MOORE, The behavior of solutions of a linear differential equation of second order, *Pacific J. Math.* **5** (1955), 125–145.

[12] R. L. POTTER, On self-adjoint differential equations of second order, *Pacific J. Math.* **3** (1953), 467–491.

[13] C. A. SWANSON, "Comparison and oscillation theory of linear differential equations", *Academic Press*, 1968.

[14] D. WILLETT, On the oscillatory behavior of the solutions of second order linear differential equations, *Ann. Polon. Math.* **21** (1969), 175–194.

[15] A. WINTNER, On the non-existence of conjugate points, *Amer. J. Math.* **73** (1951), 368–380.

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