Note on orthogonal polynomials related to theta functions.

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1. Put

$$H_n(x) = H_n(x, q) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} x^r, \tag{1.1}$$

$$G_n(x) = G_n(x, q) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} q^{r(r-n)} x^r, \tag{1.2}$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{(1-q^n)(1-p^{n-1})\cdots(1-q^{n-r+1})}{(1-q)(1-q^2)\cdots(1-q^r)}, \qquad \begin{bmatrix} n \\ 0 \end{bmatrix} = 1.$$

SZEGŐ [2] showed that

$$\frac{1}{2\pi}\int_0^{2\pi} H_m(-q^{-\frac{1}{2}} e^{i\psi}) H_n(-q^{-\frac{1}{2}} e^{-i\psi}) f(\psi) d\psi = q^{-n}(q)_n \delta_{mn}, \tag{1.3}$$

where

$$f(\psi) = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} e^{ni\psi} = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} \cos n\psi \qquad (|q| < 1) \tag{1.4}$$

and

$$(q)_n = (1-q)(1-q^2)\cdots(1-q^n), \qquad (q)_0 = 1.$$

WIGERT [3] proved that

$$\int_0^{\infty} G_m(-q^{m+\frac{1}{2}} x) G_n(-q^{n+\frac{1}{2}} x) p(x) dx = q^{-n-\frac{1}{2}}(q)_n \delta_{mn}, \tag{1.5}$$

where

$$p(x) = k\pi^{-\frac{1}{2}} \exp(-k^2 \log^2 x), \tag{1.6}$$

and

$$2k^2 = -\frac{1}{\log q} \qquad (0 < q < 1). \tag{1.7}$$

For some additional properties of $H_n(x)$ and $G_n(x)$ see [1].

The object of the present note is to prove (1.3) and (1.5) as well as some related results in a uniform and somewhat simpler way.

2. We first prove a preliminary result. Put

$$a_{mn}=\sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^{r+s}\begin{bmatrix} m \\ r \end{bmatrix}\begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}r(r-1)+\frac{1}{2}s(s-1)-rs} \tag{2. 1}$$

Then

$$a_{mn}=q^{-n}(q)_n\delta_{mn}. \tag{2. 2}$$

We recall the familiar indentity

$$\prod_{s=0}^{n-1}(1-q^s x)=\sum_{s=0}^{n}(-1)^s\begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}s(s-1)}x^s. \tag{2. 3}$$

Thus (2. 1) becomes

$$a_{mn}=\sum_{r=0}^{m}(-1)^r\begin{bmatrix} m \\ r \end{bmatrix} q^{\frac{1}{2}r(r-1)}\prod_{s=0}^{n-1}(1-q^{s-r}). \tag{2. 4}$$

There is no loss in generality in assuming $m\leqq n$. For $m<n$ it is evident that the product in the right member of (2. 4) will vanish for all r; hence

$$a_{mn}=0 \qquad (m\neq n). \tag{2. 5}$$

For $m=n$, on the other hand, the term $r=m$ yields

$$a_{nn}=(-1)^n q^{\frac{1}{2}n(n-1)}\prod_{s=0}^{n-1}(1-q^{s-n})=q^{-n}\prod_{s=1}^{n}(1-q^s)=q^{-n}(q)_n. \tag{2. 6}$$

Combining (2. 5) and (2. 6), we have proved (2. 2).

If in (2. 1) we replace q by q^{-1} we find that

$$\begin{aligned} a'_{mn}&=\sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^{r+s}\begin{bmatrix} m \\ r \end{bmatrix}\begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}r(r+1)+\frac{1}{2}s(s+1)-rm-sn+rs}= \\ &=(-1)^n q^{-\frac{1}{2}n(n-1)}(q)_n\delta_{mn}. \end{aligned} \tag{2. 7}$$

Other variants are obtained by replacing r by $m-r$ or s by $n-s$ in (2. 1). We get

$$\begin{aligned} b_{mn}&=\sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^{r+s}\begin{bmatrix} m \\ r \end{bmatrix}\begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}r(r+1)+\frac{1}{2}s(s-1)-m(r+s)+rs}= \\ &=(-1)^n q^{-\frac{1}{2}n(n+1)}(q)_n\delta_{mn}, \end{aligned} \tag{2. 8}$$

$$\begin{aligned} c_{mn}&=\sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^{r+s}\begin{bmatrix} m \\ r \end{bmatrix}\begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}r(r-1)+\frac{1}{2}s(s+1)-n(r+s)+rs}= \\ &=(-1)^n q^{-\frac{1}{2}n(n+1)}(q)_n\delta_{mn}, \end{aligned} \tag{2. 9}$$

$$\begin{aligned} d_{mn}&=\sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^{r+s}\begin{bmatrix} m \\ r \end{bmatrix}\begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}r(r+1)+\frac{1}{2}s(s+1)-(m-n)(r-s)-rs}= \\ &=(q)_n\delta_{mn}. \end{aligned} \tag{2. 10}$$

3. To prove (1. 3) we use (1. 1) and (1. 4). Then

$$\int_0^{2\pi} H_m(-q^{-\frac{1}{2}}e^{i\psi})H_n(-q^{-\frac{1}{2}}e^{-i\psi})f(\psi)d\psi =$$

$$=\sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^{r+s}\begin{bmatrix} m \\ r \end{bmatrix}\begin{bmatrix} n \\ s \end{bmatrix} q^{-\frac{1}{2}(r+s)} \sum_{t=-\infty}^{\infty} q^{\frac{1}{2}t^2} \int_0^{2\pi} e^{(r-s+t)i\psi} d\psi =$$

$$=2\pi\sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^{r+s}\begin{bmatrix} m \\ r \end{bmatrix}\begin{bmatrix} n \\ s \end{bmatrix} q^{-\frac{1}{2}(r+1)+\frac{1}{2}(r-s)^2} =$$

$$=2\pi\sum_{r=0}^{m}\sum_{s=0}^{n}(-1)^{r+s}\begin{bmatrix} m \\ r \end{bmatrix}\begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}(r-1)+\frac{1}{2}s(s-1)-rs} =$$

$$=2\pi a_{mn} = 2\pi q^{-n}(q)_n \delta_{mn},$$

where at the last step we use (2. 1) and (2. 2).

This evidently completes the proof of (1. 3).

4. To prove (1. 5) we require the following formula, which is easily verified:

$$\int_0^{\infty} x^n p(x)dx = q^{-\frac{1}{2}(n+1)^2}. \tag{4. 1}$$

This formula holds for all integral n.

Using (1. 2) and (4. 1) we get

$$\int_0^{\infty} G_n(-q^{m+\frac{1}{2}}x)G_n(-q^{n+\frac{1}{2}}x)p(x)dx =$$

$$=\sum_{r,s}(-1)^{r+s}\begin{bmatrix} m \\ r \end{bmatrix}\begin{bmatrix} n \\ s \end{bmatrix} q^{r^2+s^2+\frac{1}{2}(r+s)} \int_0^{\infty} x^{r+s}p(x)dx =$$

$$=\sum_{r,s}(-1)^{r+s}\begin{bmatrix} m \\ r \end{bmatrix}\begin{bmatrix} n \\ s \end{bmatrix} q^{r^2+s^2+\frac{1}{2}(r+s)} q^{-\frac{1}{2}(r+s+1)^2} =$$

$$=q^{-\frac{1}{2}}\sum_{r,s}(-1)^{r+s}\begin{bmatrix} m \\ r \end{bmatrix}\begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}r(r-1)+\frac{1}{2}s(s-1)-rs} =$$

$$=q^{-\frac{1}{2}}a_{mn} = q^{-n-\frac{1}{2}}(q)_n \delta_{mn},$$

which proves (1. 5).

5. If in place of (2. 2) we make use of one of the formulas (2. 7) to (2. 10), we obtain a number of results similar to (1. 3) and (1. 5). Thus by means of (2. 7) we get

$$\frac{1}{2\pi}\int_0^{2\pi} H_m(-q^{-m+\frac{1}{2}}e^{i\psi})H_n(-q^{-n+\frac{1}{2}}e^{i\psi})f(\psi)d\psi = \\ =(-1)^n q^{-\frac{1}{2}n(n-1)}(q)_n\delta_{mn}, \tag{5. 1}$$

$$\int_0^\infty G_m(-x)G_n(-qx^{-1})p(x)dx=(-1)^n q^{-\frac{1}{2}(n^2-n+1)}(q)_n\delta_{mn}. \tag{5. 2}$$

Similarly use of (2. 8) gives

$$\frac{1}{2\pi}\int_0^{2\pi} H_m(-q^{-m+\frac{1}{2}}e^{i\psi})H_n(-q^{-m-\frac{1}{2}}e^{i\psi})f(\psi)d\psi = \\ =(-1)^n q^{-\frac{1}{2}n(n+1)}(q)_n\delta_{mn}, \tag{5. 3}$$

$$\int_0^\infty G_m(-x)G_n(-x^{-1})p(x)dx=(-1)^n q^{-\frac{1}{2}(n^2+n+1)}(q)_n\delta_{mn}, \tag{5. 4}$$

while (2. 9) yields

$$\frac{1}{2\pi}\int_0^{2\pi} H_m(-q^{-n-\frac{1}{2}}e^{i\psi})H_n(-q^{-n+\frac{1}{2}}e^{i\psi})f(\psi)d\psi = \\ =(-1)^n q^{-\frac{1}{2}n(n+1)}(q)_n\delta_{mn}, \tag{5. 5}$$

$$\int_0^\infty G_m(-x^{-1})G_n(-x)p(x)dx=(-1)^n q^{-\frac{1}{2}(n^2+n+1)}(q)_n\delta_{mn}. \tag{5. 6}$$

We note that (5. 3) and (5. 5) are equivalent; also (5. 4) and (5. 6) are equivalent.

Finally by means of (2. 10) we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} H_m(-q^{n-m+\frac{1}{2}}e^{i\psi})H_n(-q^{m-n+\frac{1}{2}}e^{-i\psi})f(\psi)d\psi=(q)_n\delta_{mn}. \tag{5. 7}$$

$$\int_0^\infty G_m(-q^{n+\frac{3}{2}}x)G_n(-q^{m+\frac{3}{2}}x)p(x)dx=q^{-\frac{1}{2}}(q)_n\delta_{mn}. \tag{5. 8}$$

If in place of (1. 4) we employ

$$f_1(\psi)=\sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n(n-1)}e^{ni\psi}, \tag{5.9}$$

it is easily verified that

$$\frac{1}{2\pi}\int_0^{2\pi} H_m(-e^{i\psi})H_n(-q^{-1}e^{-ni\psi})f_1(\psi)\,d\psi=q^{-n}(q)_n\delta_{mn}. \tag{5.10}$$

Similarly we can find variants of (5. 1), (5. 3) and (5. 7). Variants of (1. 5), (5. 2), (5. 4), (5. 8) are obtained by using in place of (1. 6) the weight function

$$p_1(x)=k\pi^{-\frac{1}{2}}\exp\left\{-\left(k\log x+\frac{1}{4k}\right)^2\right\}=kq^{\frac{1}{2}}(\pi x)^{-\frac{1}{2}}p(x)$$

which satisfies

$$\int_0^{\infty} x^n p_1(x)\,dx=q^{-\frac{1}{2}n(n+1)}$$

6. Since

$$\sum_{n=0}^{\infty} H_n(x)\frac{t_n}{(q)_n}=\prod_{n=0}^{\infty}(1-q^nt)^{-1}(1-q^nxt)^{-1} \qquad (|t|<1)$$

and

$$\sum_{-\infty}^{\infty} q^{\frac{1}{2}n^2}z^n=\prod_{n=1}^{\infty}(1-q^n)(1-q^{n-1}z)(1-q^{n-1}z^{-1}),$$

it is easily verified that (1. 3) implies

$$\frac{1}{2\pi}\int_0^{2\pi}\prod_{n=0}^{\infty}\frac{(1-q^ne^{i\psi})(1-q^ne^{-i\psi})}{(1+q^{n+\frac{1}{2}}e^{i\psi}t)(1+q^{n+\frac{1}{2}}e^{-i\psi}z)}\,d\psi= \tag{6.1}$$
$$=\prod_{n=1}^{\infty}\frac{(1-q^nt)(1-q^nz)}{(1-q^n)(1-q^ntz)}.$$

Similarly since

$$\sum_{n=0}^{\infty}(-1)^nq^{\frac{1}{2}n(n-1)}G_n(x)\frac{t^n}{(q)_n}=\prod_{n=0}^{\infty}(1-q^nt)(1-q^nxt),$$

(5. 4) is equivalent to

$$\int_0^{\infty}\prod_{n=1}^{\infty}(1+q^nxt)(1+q^nx^{-1}z)\cdot p(x)\,dx=q^{-\frac{1}{2}}\prod_{n=1}^{\infty}\frac{1-q^ntz}{(1-q^nt)(1-q^nz)}. \tag{6.2}$$

7. The following formulas are of a slightly different kind. For brevity put

$$(x)_n = \prod_{s=0}^{n-1}(1-q^s x),$$

the left member of (2. 3). Then we have

$$\frac{1}{2\pi}\int_0^{2\pi} H_n(xe^{i\psi})f(\psi)d\psi = (-xq^{\frac{1}{2}})_n, \tag{7. 1}$$

$$\frac{1}{2\pi}\int_0^{2\pi} (xe^{i\psi})_n f(\psi)d\psi = G_n(-q^{n-\frac{1}{2}}x), \tag{7. 2}$$

$$\int_0^{\infty} G_n(xz)p(z)dz = q^{-\frac{1}{2}}(-xq^{-n-\frac{1}{2}})_n, \tag{7. 3}$$

$$\int_0^{\infty} G_n(xz)p_1(z)dz = (-xq^{-n})_n, \tag{7. 3)'}$$

$$\int_0^{\infty} (xz)_n p(z)dz = q^{-\frac{1}{2}}H_n(-xq^{-\frac{3}{2}}), \tag{7. 4}$$

$$\int_0^{\infty} (xz)_n p_1(z)dz = H_n(-xq^{-1}), \tag{7. 4)'}$$

where in (7. 3)' and (7. 4)', $p_1(z)$ has the same meaning as in (5. 11).

Finally if

$$f(\psi, q^2) = \sum_{n=-\infty}^{\infty} q^{n^2}e^{ni\psi},$$

$$p(x, q^2) = k(2\pi)^{\frac{1}{2}}\exp\left(-\frac{1}{2}k^2\log^2 x\right),$$

where k is defined by (1. 7), then we have also

$$\frac{1}{2\pi}\int_0^{2\pi} H_n(xe^{i\psi})f(\psi, q^2)d\psi = G_n(xq^n), \tag{7. 5}$$

$$\int_0^{\infty} G_n(xz)p(z, q^2)dz = q^{-1}H_n(xq^{-n-2}). \tag{7. 6}$$

The proof of these formulas is immediate.

We remark that if

$$f(\psi, z, q^2) = \sum_{n=-\infty}^{\infty} q^{n^2} z^{ni} e^{ni\psi},$$

then

$$\frac{1}{2\pi}\int_0^{2\pi} H_n(xe^{i\psi}) f(\psi, z, q^2)\, d\psi = G_n(xz^{-1}q^n). \tag{7. 7}$$

The formulas (7. 1) and (7. 2) can be extended in a similar manner.

Bibliography.

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