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Common fixed points of compatible set-valued mappings

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Abstract. In a recent paper IMDAD and AHMAD [1] proved several fixed point theorems for set-valued mappings satisfying conditions weaker than commuting, such as weakly commuting, quasi-commuting, and slightly commuting. In this paper we show that two of these definitions are special cases of δ-compatibility, and prove a fixed point theorem for four maps satisfying the δ-compatibility condition.

Let $(X,\ d)$ be a complete metric space, $B(X)$ the collection of all nonempty bounded subsets of X. For $A, B, \in B(X)$, define $\delta(A,\ B) = \sup\,\{d(a,b) : a \in A,\ b \in B\}$. Let F be a multivalued map from X into $B(X)$, I a single-valued selfmap of X. From [4] and [2], the pair (F, I) is weakly commuting on X if for any x in X,

$$\delta(FIx,\ IFx) \le \max\,\{\delta(Ix, Fx),\ \operatorname{diam} IFx\}$$

quasi-commuting on X if, for any x in X $IFx \subseteq FIx$,

slightly commuting on X if for any x in X,

$$\delta(FIx,\ IFx) \le \max\,\{\delta(Ix, Fx),\ \operatorname{diam} Fx\}.$$

From [3] the pair (F, I) is δ-compatible if $\delta(IFx_n, FIx_n) \to 0$ whenever $\{x_n\}$ is a sequence in x such that $Ix_n \to t$ and $Fx_n \to \{t\}$ for some $t \in X$. As noted in [3], weakly commuting and slightly commuting imply δ-compatibility.

Let Φ be the set of all real-valued functions $\phi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ which are semi-continuous from the right and nondecreasing in each of the coordinate variables such that $\phi(t, t, t, at, bt) < t$ for each $t > 0,\ a, b \ge 0,\ a + b \le 4$.

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Let Ψ denote the set of real-valued functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ which are upper semi-continuous from the right and nondecreasing with $\psi(t) < t$ for $t > 0$.

Theorem. *Let $F, G : X \to B(X)$, I, J two selfmaps of X. Suppose that (F, I) and (G, J) are δ-compatible, one of the four maps is continuous, $F(X) \subseteq J(X)$, $G(X) \subseteq I(X)$, and $\phi \in \Phi$. Suppose that*

$$\begin{aligned} &\delta(Fx, Gy) \\ &\quad \le \phi(\delta(Ix, Fx),\ \delta(Jy, Gy),\ \delta(Ix, Gy),\ \delta(Jy, Fx),\ d(Ix, Jy)) \end{aligned} \tag{1}$$

$\psi \in \Psi$, satisfies, for $t > 0$, $a, b > 0$, $a + b \le 4$,

$$\psi(t) = \max \left\{ \begin{array}{lll} \phi(t, t, t, at, bt), & \phi(t, 0, 0, t, 0), & \phi(0, 0, t, t, t), \\ \phi(0, t, t, 0, 0) & & \end{array} \right\} < t.$$

Then F, G, I and J have a unique common fixed point z such that $Iz = Jz = z$ and $Fz = Gz = \{z\}$. Also, z is the unique common fixed point of F and I, and of G and J; i.e., $z = Iz = Jz = Fz = Gz$.

Proof. Let $x_0 \in X$, y_1 an arbitrarily chosen point in $X_1 := Fx_0$. Since $F(X) \subseteq J(X)$, there exists a point $x_1 \in X$ such that $Jx_1 = y_1$. Choose an arbitrary point y_2 in $X_2 := Gx_1$. Since $G(X) \subseteq I(X)$, there exists a point $x_2 \in X$ with $Ix_2 = y_2$. In general, for any $x_{2n} \in X$ with $y_{2n+1} \in X_{2n+1} := Fx_{2n}$, there exists an $x_{2n+1} \in X$ satisfying $Jx_{2n+1} = y_{2n+1}$. For $y_{2n+2} \in X_{2n+2} := Gx_{2n-1}$, there exists an $x_{2n+2} \in X$ such that $Ix_{2n+2} = y_{2n+2}$. Set $V_n := \delta(X_n,\ X_{n+1})$.

Case 1. Suppose there exists an n for which $V_n = 0$. Then, if n is even we have $\delta(Gx_{2n-1},\ Fx_{2n}) = 0$. This implies that $Fx_{2n} = y_{2n+1} = Jx_{2n+1} = Gx_{2n+1} = y_{2n+2} = Ix_{2n+2}$. Since G and J are δ-compatible, from Proposition 3.1 of [3],

$$GJx_{2n+1} = JGx_{2n+1} = CGx_{2n+1}. \tag{2}$$

From (1),

$$\begin{aligned} \delta(Fx_{2n+2}, Gx_{2n+1}) &\le \phi(\delta(Fx_{2n+2}, Gx_{2n+1}), 0, 0, \delta(Fx_{2n+2}, Gx_{2n+1}), 0) \\ &\le \psi(\delta(Fx_{2n+2}, Gx_{2n+1}), 0, 0, \delta(Fx_{2n+2}, Gx_{2n+1}), 0) \\ &< \delta(Fx_{2n+2}, Gx_{2n+1}) \end{aligned}$$

which implies that $Fx_{2n+2} = Gx_{2n+1}$. From the δ-compatibility of F and I,

$$IFx_{2n+2} = FIx_{2n+2} = FFx_{2n+2}. \tag{3}$$

Again from (1),

$$\begin{aligned}\delta(FFx_{2n+2},\ Fx_{2n+2}) &= \delta(FFx_{2n+2},\ Gx_{2n+1})\\ &\leq \phi(0,0,\ \delta(FFx_{2n+2},\ Fx_{2n+2}),\ \delta(FFx_{2n+2},\ Fx_{2n+2}),\\ &\qquad \delta(FFx_{2n+2},\ Fx_{2n+2}))\\ &\leq \psi(0,0,\ \delta(FFx_{2n+2},\ Fx_{2n+2}),\ \delta(FFx_{2n+2},\ Fx_{2n+2}),\\ &\qquad \delta(FFx_{2n+2},\ Fx_{2n+2}))\\ &< \delta(FFx_{2n+2},\ Fx_{2n+2}),\end{aligned}$$

and $FFx_{2n+2} = Fx_{2n+2}$. Therefore Fx_{2n+2} is a fixed point of F. Using (3), Fx_{2n+2} is a fixed point of I. Since $Fx_{2n+2} = Gx_{2n+1}$, using (1),

$$\begin{aligned}\delta(Gx_{2n+1},\ GGx_{2n+1}) &= \delta(Gx_{2n+2},\ GGx_{2n+1})\\ &\leq \phi(0,0,\ \delta(Gx_{2n+1},\ GGx_{2n+1}),\ \delta(Gx_{2n+1},\ GGx_{2n+1}),\\ &\qquad \delta(Gx_{2n+1},\ GGx_{2n+1}))\\ &\leq \psi(0,0,\ \delta(Gx_{2n+1},\ GGx_{2n+1}),\ \delta(Gx_{2n+1},\ GGx_{2n+1}),\\ &\qquad \delta(Gx_{2n+1},\ GGx_{2n+1}))\end{aligned}$$

which yields that $Gx_{2n+1} = GGx_{2n+1}$, and hence Gx_{2n+1} is a fixed point of G and, from (2), a fixed point of J. Thus Fx_{2n+2} is a common fixed point of F, G, I, and J.

Case 2. Suppose that $V_n > 0$ for all n. Then, using the same argument as in [2], it follows that $\{y_n\}$ is a Cauchy sequence and hence converges to a point z in X. Thus $\lim y_{2n} = \lim Ix_{2n} = \lim y_{2n+1} = \lim Jx_{2n+1} = z$ and $\lim Fx_{2n} = \lim Gx_{2n+1} = \{z\}$. From Proposition 3.1 of [3], $\lim \delta(FIx_{2n},\ IFx_{2n}) = 0$.

Suppose that I is continuous. Then $\lim Iy_{2n} = Iz$, and, from the argument in [2], $Iz = z,\ Fz = \{z\}$, and $Gz' = \{z\},\ Jz' = z$. Since G and J are δ-compatible, it follows that $GJz' = JGz'$, and hence that $Gz = Jz$, which leads to z being a common fixed point of the four maps.

Suppose now that F is continuous. Then, as in [2], $\lim\ Fy_{2n} = Fz$, and $Fz = \{z\}$. Since $F(X) \subseteq J(X)$, there exists a point z' in X such that $Jz' = z$. Applying (1) to $\delta(Gz',\ Fx_{2n})$ and then taking the limit as $n \to \infty$ it follows that $Gz' = \{z\}$. Since J and G are δ-compatible, $GJz' = JGz'$, which leads to $Gz = Jz$. Applying (1) to $\delta(Fx_{2n},\ Gz)$ and letting $n \to \infty$, we obtain that $Gz = \{z\}$. Thus $Jz = Gz = \{z\}$.

Since $G(X) \subseteq I(X)$, there exists a point z'' in X such that $Iz'' = z$. Thus

$$\begin{aligned}\delta(Fz'', z) = {} & \delta(Fz'',\ Gz) \le \phi(\delta(Fz'', z),\ 0, 0,\ \delta(Fz'', z),\ 0) \\ \le {} & \psi(\delta(Fz'',\ z),\ 0, 0,\ \delta(Fz'', z),\ 0) \le \delta(Fz'', z),\end{aligned}$$

which implies that $Fz'' = \{z\}$. Using the fact that F and I are δ-compatible it follows that $Fz = Iz$. Therefore z is a common fixed point of the four maps.

The proofs for J or G continuous are similar and will be omitted.

The uniqueness of z follows from (1). □

Theorems 3.1–3.3 of [2] are special cases of the theorem of this paper.

We remark that Theorem 1 of [1] can also be extended to compatible maps.

References

[1] A. Ahmad and M. Imdad, A common fixed point theorem for four mappings satisfying a rational inequality, *Publ. Math. Debrecen* **41** (1992), 181–187.

[2] M. Imdad and A. Ahmad, On common fixed point of mappings and set-valued mappings with some weak conditions of commutativity, *Publ. Math. Debrecen* **44** (1994), 105–114.

[3] G. Jungck and B. E. Rhoades, Some fixed point theorems for compatible maps, *Internat. J. Math. & Math. Sci.* **16** (1993), 417–428.

[4] S. Sessa, M. S. Khan and M. Imdad, A common fixed point theorem with a weak commutativity condition, *Glasnik Mat.* **21** (1986), 225–235.

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