A note on countable compact spaces

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Abstract: In this note we prove that in the upper limit topology of any ordered set, every compact subset must be well-ordered. Assuming a well-known topological characterisation of the space of all rational numbers, the above result leads to a simpler proof of a theorem of Mazurkiewicz and Sierpinski which characterises all countable compact Hausdorff spaces.

Let X be any set endowed with a total order $\leqq$. For each a in X, let $X_{(a}=\{x\in X|x>a\}$ and let $X_{a]}=\{x\in X|x\leqq a\}$. Then the topology on X for which $\{X_{(a}|a\in X\}U\{X_{a]}|a\in X\}$ is a sub-base, is called the upper limit topology on X. It is always a zero-dimensional Hausdorff topology finer than the order topology. We shall now show that all of its compact subsets must be well-ordered.

Theorem. *Let X be an ordered set with upper limit topology. Let $K\subset X$ be compact. Then K is well-ordered in the induced order. Further, the relative topology on K coincides with its order topology.*

PROOF. Let A be any non-empty subset of K. We claim that A has a least element. Now consider $\bigcup_{a\in A} X_{(a}$. If x is an element of X not belonging to this union, then x is either the least element of A or x is less than every element of A. In the latter case $X_{x]}$ is an open set containing x disjoint with A. Hence if A had no least element, it would follow that $\bar{A}\subset\bigcup_{a\in A} X_{(a}$. But since $\bar{A}$ is compact, this implies that there exist a finite number of elements $\{a_1, a_2, \dots, a_n\}$ in A such that $\bar{A}\subset\bigcup_{i=1}^{n} X_{(a_i}$. Let now $a_0=\{\min a_1, a_2, \dots, a_n\}$. Then $a_0\in A$ and $\bar{A}\subset X_{(a_0}$. It follows that a_0 ist the least element of A. This completes the proof that K is well-ordered.

Now let $a\in K$. Let $K_{(a}=\{x\in K/x>a\}$ and $K_{a)}=\{x\in K/x<a\}$. Then $K_{(a}$'s and $K_{a)}$'s form a base for the order topology of K. Now we have for each a in K.

(i) $K_{(a}=K\cap X_{(a}$.

(ii) If there exists a largest element b in K which is strictly less than a, then

$$K_{a)}=K_{b]}$$

(iii) If among the elements of K that are strictly less than a, there is nor largest element, then

$$K_{a)}=\bigcup_{\substack{b\in K\\ b<a}} K_{b]}$$

These there facts above that the order topology of K is coarser than the relative topology of K from the upper limit topology of X. But since K is compact in the upper limit topology, the two topologies coincide.

Corollary 1. In the upper limit topology of the real line, each compact subset is countable and well-ordered. This gives a different proof of the known fact that this upper limit topology is not σ-compact.

Corollary 2. (Mazurkiewicz and Sierpinski): Every countable compact Hausdorff space is homeomorphic to the well-ordered space $[1, \alpha]$ for some countable ordinal α.

Proof. Let X be any countable compact Hausdorff space. Then it is well-known that X is metrisable and hence homeomorphic to a subspace of Q, where Q is the space of all rational numbers. Consider the upper limit topology of Q. It is obviously regular. It is first countable and hence second countable, since the space is countable. Therefore, by Urysohn's metrisation theorem, it is metrisable. Also, it has no isolated points. But an earlier result of Sierpinski [2] says that every countably infinite perfect metric space must be homeomorphic to the usual topology of Q. Therefore on Q, the upper limit topology and the usual topology are homeomorphic. It follows that X is homeomorphic to a compact subspace of the upper limit topology of Q. Therefore by the above theorem, X is well ordered and hence the result.

Remark. The original proof of Corollary 2 given in [1] uses an ordinal invariant and the principle of transfinite induction. We have shown that the major result of [1] is a consequence of the result of [2]. It may be noted that [1] and [2] have appeared together. with [2] just preceding [1].

References

[1] S. Mazurkiewicz and W. Sierpinski, Contribution a la topologie des ensembles denombrables, *Fund. Math.* **1** (1920), 17—27.

[2] W. Sierpinski, Sur une propriete topologique des ensembles denombrables denses en soi. *Fund. Math.* **1** (1920), 11—16.

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