On some properties of Finsler spaces based on the indicatrices

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Introduction

For the study of Finsler spaces, the indicatrices play an important role. In the present paper, we shall deal with some special Finsler spaces characterized by Riemannian geometric properties of the indicatrices.

In § 1, we shall state the fundamental formulas. In § 2, we deal with a Finsler space such that all the indicatrices are conformally flat. In § 3, we obtain a necessary and sufficient condition for the vector field C_α[1]) (the tangential component of the torsion vector C_i to the indicatrix) on the indicatrix is conformal Killing. Moreover, as an application of the result, we shall deal with the M. MATSUMOTO's conjecture on a Finsler space satisfying T-condition [4].[2]) Finally, in § 4, we shall deal with C-reducible Finsler spaces.

The terminologies and notations in the present paper are referred to M. MATSUMOTO's monograph [6].

§ 1. *Fundamental formulas*

Let (M^n, L) be an n-dimensional Finsler space, where M^n is the underlying n-dimensional manifold and $L=L(x, y)$ is the Finsler fundamental function. L being a differentiable function of the point $x=(x^i)$ and the element of support $y=(y^i)\in T_x(M^n)$ and is positively homogeneous of degree one with respect to y, where $T_x(M^n)$ is the tangent space of M^n at the point x.

Now, we shall restrict our discussion to the fixed point $x_0(\in M^n)$ and the corresponding tangent space $T_{x_0}(M^n)$ of M^n at the point x_0. Then the space $V^n=T_{x_0}(M^n)-\{0\}$ being an n-dimensional Riemannian space with the metric tensor $g_{ij}(x_0, y)=\frac{1}{2}\partial^2 L^2/\partial y^i\partial y^j$, where 0 is the origin of $T_{x_0}(M^n)$. It is easily seen that the Riemannian connection and the curvature tensor of V^n are the $h(hv)$-torsion tensor (the Cartan torsion tensor) C_{ijk} and the v-curvature tensor (the first curvature tensor of the Finsler space) S_{ijkl}, respectively. And the covariant differentiation of the tensor T_{ij} on V^n is the same as the v-covariant differentiation: $T_{ij}|_k=\partial T_{ij}/\partial y^k-T_{tj}C^t_{i\,k}-T_{it}C_j{}^t{}_k$.

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[1]) Greek indices run over the range $\{1, 2, \ldots, n-1\}$ and Roman indices run over the range $\{1, 2, \ldots, n\}$.

[2]) Numbers in brackets refer to the references at the end of the paper.

Moreover, it is well known that

(1.1a) $$C_{ijk}y^i = C_{ijk}y^j = C_{ijk}y^k = 0,$$

(1.1b) $$S_{ijkl}y^i = S_{ijkl}y^j = S_{ijkl}y^k = S_{ijkl}y^l = 0,$$

(1.1c) $$L|_i = l_i,$$

(1.1d) $$l_i|_j = L^{-1}h_{ij},$$

(1.1e) $$h_{ij}|_k = -L^{-1}(h_{ik}l_j + h_{jk}l_i),$$

where $l_i = y_i/L$ and $h_{ij} = g_{ij} - l_i l_j$.

The indicatrix I_{x_0} at $x_0(\in M^n)$ is defined by the hypersurface which is given by the following equation

(1.2) $$g_{ij}(x_0, y)\,y^i y^j = 1, \quad \big(\text{or } L(x_0, y) = 1\big).$$

Then I_{x_0} is locally represented by the equation $y^i = y^i(u^\alpha)$. We shall denote the projection factor $\partial y^i/\partial u^\alpha$ and the unit normal vector to I_{x_0} by $B_\alpha{}^i$ and N^i, respectively. Then the induced Riemannian metric tensor $g_{\alpha\beta}$ of I_{x_0} is given by $g_{\alpha\beta} = g_{ij}B_\alpha{}^i B_\beta{}^j$.

We shall denote the operation of D-symbol due to Van der Wearden—Bortolotti by D_α. Then it is well known that

(1.3a) $D_\alpha g_{ij} = 0,$ (1.3b) $D_\gamma g_{\alpha\beta} = 0,$

(1.3c) $D_\alpha B_\beta{}^i = b_{\alpha\beta}N^i,$ (1.3d) $D_\alpha N^i = -g^{\beta\gamma}b_{\alpha\gamma}B_\beta{}^i,$

where the symmetric tensor $b_{\alpha\beta}$ is the second fundamental tensor of I_{x_0} and $g^{\alpha\beta}$ are the contravariant components of $g_{\alpha\beta}$. Moreover, from (1.1a) it is clear that

(1.4) $$D_\alpha y^i = B_\alpha{}^i.$$

Applying D_α to (1.2) and using (1.3a) and (1.4), we obtain that $g_{ij}B_\alpha{}^i y^j = 0$. This equation gives us that the normal vector $N^i(u)$ is identical with $y^i(u)$. On the other hand, the indicatrix I_{x_0} is totally umbilical and the first mean curvature of I_{x_0} is equal to -1 identically. Then we have $b_{\alpha\beta} = -g_{\alpha\beta}$. This result gives us that

(1.5) $$D_\alpha B_\beta{}^i = -g_{\alpha\beta}y^i.$$

Let $R_{\alpha\beta\gamma\delta}$ be the curvature tensor of I_{x_0}, then from the Gauss equation, we have that

(1.6) $$R_{\alpha\beta\gamma\delta} = S_{ijkl}B_\alpha{}^i B_\beta{}^j B_\gamma{}^k B_\delta{}^l - g_{\alpha\delta}g_{\beta\gamma} + g_{\alpha\gamma}g_{\beta\delta}.$$

On the other hand, from the definitions of h_{ij} and $g_{\alpha\beta}$ we have

(1.7) $$g_{\alpha\beta} = h_{ij}B_\alpha{}^i B_\beta{}^j.$$

Substituting (1.7) in (1.6), we get

(1.8) $$R_{\alpha\beta\gamma\delta} = (S_{ijkl} - h_{il}h_{jk} + h_{ik}h_{jl})B_\alpha{}^i B_\beta{}^j B_\gamma{}^k B_\delta{}^l.$$

Contracting (1.8) by $g^{\alpha\delta}$, we obtain

(1.9) $$R_{\beta\gamma} = \big(S_{jk} - (n-2)h_{jk}\big)B_\beta{}^i B_\gamma{}^k,$$

where $R_{\beta\gamma}$ is the Ricci tensor of I_{x_0} and $S_{jk}=g^{il}S_{ijkl}$ is v-Ricci tensor. Moreover contracting (1.9) by $g^{\beta\gamma}$, we have

$$R = S-(n-2)(n-1), \tag{1.10}$$

where R is the scalar curvature of I_{x_0} and $S=g^{jk}S_{jk}$ is v-scalar curvature.

§ 2. *Finsler spaces with conformally flat indicatrices*

In this section, let the dimension of the Finsler space M^n be more than three, then the dimension of the indicatrix is more than two.

Now, we assume that the indicatrix I_{x_0} is conformally flat, then from the well-known theorem of Weyl we have that

$$\begin{aligned} C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}+(R_{\alpha\gamma}g_{\beta\delta}+R_{\beta\delta}g_{\alpha\gamma}-R_{\alpha\delta}g_{\beta\gamma}-R_{\beta\gamma}g_{\alpha\delta})/(n-3)- \\ -R(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma})/(n-2)(n-3) = 0 \quad \text{for} \quad n \geqq 5, \end{aligned} \tag{2.1}$$

$$C_{\alpha\beta\gamma} = D_\alpha R_{\beta\gamma}-D_\beta R_{\alpha\gamma}-(g_{\beta\gamma}D_\alpha R-g_{\alpha\gamma}D_\beta R)/2(n-2) = 0 \quad \text{for} \quad n = 4, \tag{2.2}$$

where $C_{\alpha\beta\gamma\delta}$ is the Weyl's conformal curvature tensor.

First, we shall consider the case of $n\geqq 5$. Substituting (1.7), (1.8), (1.9) and (1.10) in (2.1), we have

$$\begin{aligned} \{S_{ijkl}+(S_{ik}h_{jl}+S_{jl}h_{ik}-S_{il}h_{jk}-S_{jk}h_{il})/(n-3)- \\ -S(h_{ik}h_{jl}-h_{il}h_{jk})/(n-2)(n-3)\}B_\alpha^{\ i}B_\beta^{\ j}B_\gamma^{\ k}B_\delta^{\ l} = 0. \end{aligned} \tag{2.3}$$

Let $Q_{ijkl}=S_{ijkl}+(S_{ik}h_{jl}+S_{jl}h_{ik}-S_{il}h_{jk}-S_{jk}h_{il})/(n-3)-S(h_{ik}h_{jl}-h_{il}h_{jk})/(n-2)(n-3)$, then from (1.1b) and $h_{ij}y^j=0$ obtained by the definition of h_{ij} we have that $Q_{ijkl}y^i=Q_{ijkl}y^j=Q_{ijkl}y^k=Q_{ijkl}y^l=0$. Therefore the following equation is satisfied on I_{x_0} by the above equation and (2.3)

$$\begin{aligned} S_{ijkl}+(S_{ik}h_{jl}+S_{jl}h_{ik}-S_{il}h_{jk}-S_{jk}h_{il})/(n-3)- \\ -S(h_{ik}h_{jl}-h_{il}h_{jk})/(n-2)(n-3) = 0. \end{aligned} \tag{2.4}$$

Since S_{ijkl} is the homogeneous tensor of degree -2 with respect to y, we are able to extend (2.4) on V^n. Then we get

$$S_{ijkl} = h_{il}C_{jk}+h_{jk}C_{il}-h_{ik}C_{jl}-h_{jl}C_{ik}, \tag{2.5}$$

where $C_{ij}=S_{ij}/(n-3)-Sh_{ij}/2(n-2)(n-3)$.

Conversely, if the v-curvature tensor satisfies (2.5) at the point x_0 of M^n, then the indicatrix I_{x_0} is conformally flat, as it has been proved by M. Matsumoto [6].

Next, we shall consider the case of $n=4$ and assume that I_x is conformally flat. For the dimension of the indicatrix I_{x_0} is three, the Weyl's conformal curvature tensor $C_{\alpha\beta\gamma\delta}$ vanishes on I_{x_0}. Therefore (2.5) is satisfied on V^n.

Now we substitute (1.7), (1.9) and (1.10) in (2.2) and use (1.5), then we have

$$\{S_{ij}|_k-S_{ik}|_j-(S|_k h_{ij}-S|_j h_{ik})/2(n-2)\}B_\alpha^{\ i}B_\beta^{\ j}B_\gamma^{\ k} = 0. \tag{2.6}$$

Let

$$Q_{ijk} = S_{ij}|_k - S_{ik}|_j + L^{-1}S_{ij}l_k - L^{-1}S_{ik}l_j - \frac{(S_k|h_{ij} - S_j|h_{ik} + 2SL^{-1}h_{ij}l_k - 2SL^{-1}h_{ik}l_j)}{2(n-2)},$$

then it is clear from (1.1b) and $S_{ij}|_t y^t = -2S_{ij}$. that $Q_{ijk}y^i = Q_{ijk}y^j = Q_{ijk}y^k = 0$ These equations and (2.6) show us that $Q_{ijk}=0$ on I_{x_0}. Therefore we have the following equation on V^n

$$S_{ij}|_k - S_{ik}|_j + L^{-1}S_{ij}l_k - L^{-1}S_{ik}l_j - \tag{2.7}$$

$$-(S|_k h_{ij} - S|_j h_{ik} + 2SL^{-1}h_{ij}l_k - 2SL^{-1}h_{ik}l_j)/2(n-2) = 0.$$

From the v-covariant differentiation of C_{ij}, (2.7) is rewritten in the form

$$C_{ij}|_k - C_{ik}|_j + L^{-1}C_{ij}l_k - L^{-1}C_{ik}l_j = 0. \tag{2.8}$$

Conversely, if (2.8) is satisfied at the point x_0 of M^n, then we easily obtain $C_{\alpha\beta\gamma}=0$ on I_{x_0} from the contraction of (2.7) by B_α^i, B_β^j and B_γ^k. These results give us

Theorem 1. *Let M^n be an n-dimensional Finsler space. A necessary and sufficient condition for all the indicatrices are conformally flat is that*

$$S_{ijkl} = h_{il}C_{jk} + h_{jk}C_{il} - h_{ik}C_{jl} - h_{jl}C_{ik} \quad \text{for} \quad n \geqq 5,$$

$$C_{ij|k} - C_{ik|j} + L^{-1}C_{ij}l_k - L^{-1}C_{ik}l_j = 0 \quad \text{for} \quad n = 4.$$

If $n \geqq 5$ and the indicatrix is conformally flat, then $C_{\alpha\beta\gamma}=0$ from $D_\delta C_\alpha{}^\delta{}_{\beta\gamma} = (n-4)C_{\alpha\beta\gamma}/(n-3)$. So we have

Corollary 2. *Let M^n be an $n(\geqq 5)$-dimensional Finsler space. If all the indicatrices are conformally flat, then we have that*

$$C_{ij}|_k - C_{ik}|_j + L^{-1}C_{ij}l_k - L^{-1}C_{ik}l_j = 0.$$

§ 3. *Indicatrices with conformal Killing vector field C_α*

Let C_i be a vector field given by $C_i = g^{jk}C_{ijk}$. Then we get a vector field $C_\alpha \doteq C_i B_\alpha^i$ on I_{x_0}. From the fact that C_{ijk} are connection coefficients of V^n we have that $C_i = \partial(\log\sqrt{g})/\partial y^i$, where g is the determinant of the metric tensor g_{ij}. Therefore C_α is the partial derivative of the function $\log\sqrt{g}$ by u^α. Accordingly, it is easy to see that $D_\alpha C_\beta = D_\beta C_\alpha$.

Now we assume that C_α is the conformal Killing vector on I_{x_0}. Then we have that

$$D_\alpha C_\beta = \varrho g_{\alpha\beta}, \tag{3.1}$$

where ϱ is a certain function on I_{x_0}. From the definition of C_β, (1.1a) and (1.7), we get

$$(C_i|_j - \varrho h_{ij})B_\alpha^i B_\beta^j = 0. \tag{3.2}$$

For the tensor $Q_{ij} \doteq C_i|_j - \varrho h_{ij} + L^{-1}C_i l_j + L^{-}C_j l_i$ it is easily seen $Q_{ij}y^i = Q_{ij}y^j = 0$. From these equations and (3.2) we have $Q_{ij}=0$ on I_{x_0}. This result gives us the following equation

$$C_i|_j = \varrho h_{ij} - L^{-1} C_i l_j - L^{-1} C_j l_i \tag{3.3}$$

at the point x_0 of M^n.

Conversely, if (3.3) is satisfied at the point x_0 of M^n, it is seen that $D_\alpha C_\beta = \varrho g_{\alpha\beta}$. Therefore C_α is the conformal Killing vector on I_{x_0}. So we have that

Theorem 3. *A necessary and sufficient condition for the vector C_α is conformal Killing vector is that*

$$C_i|_j = \varrho h_{ij} - L^{-1} C_i l_j - L^{-1} C_j l_i.$$

Now we assume that M^n satisfies the condition (3.3). Then, by Theorem 3, C_α is a conformal Killing vector: $D_\beta C_\alpha = \varrho g_{\alpha\beta}$ on I_{x_0}. Moreover, we shall assume that the indicatrix I_{x_0} is closed and ϱ has a constant sign. Then we have $g^{\alpha\beta}D_\beta C_\alpha = (n-1)\varrho \geqq 0$ (or $\leqq 0$). Accordingly, we get that $\log \sqrt{g}$ is constant on I_{x_0}. On the other hand, $\log \sqrt{g}$ is homogeneous of degree zero with respect to y. So we obtain that $\log \sqrt{g}$ is constant on V^n. Then we have $C_i = 0$ at the point x_0 of M^n. Accordingly, by Deike's theorem [1], we have M^n is a Riemannian space. Therefore we have

Theorem 4. *Let M^n be an n-dimensional Finsler space and assume that all the indicatrices I_x are closed. If M^n satisfies a condition: $C_{i|j} = \varrho h_{ij} - L^{-1}C_i l_j - L^{-1}C_j l_i$ and ϱ has the constant sign, then M^n is a Riemannian space.*

Now we shall show two corollaries of Theorem 4.

Corollary 5. (Matsumoto's *conjecture*) *Let M^n be an n-dimensional Finslerspace and assume that all the indicatrices are closed. If M^n satisfies the T-condition, then M^n is a Riemannian space.*

Proof. If M^n satisfies the T-condition, then by the definition we have

$$T_{ijkl} = C_{ijk}|_l + L^{-1} C_{ijk} l_l + L^{-1} C_{ljk} l_i + L^{-1} C_{ilk} l_j + L^{-1} C_{ijl} l_k = 0. \tag{3.4}$$

Contracting (3.4) by g^{ij}, we have that $C_k|_l = -L^{-1}C_i l_j - L^{-1}C_j l_i$. This equation is the case of $\varrho = 0$ of (3.3).

Corollary 6. *Let M^n be an n-dimensional Finsler space and assume that all the indicatrices are closed. If M^n satisfies $C_i|_j = \varrho h_{ij} - L^{-1}C_i l_j - L^{-1}C_j l_i$ and the function $L^2 C_t C^t$ is a function of the point x only on M^n, then M^n is a Riemannian space.*

Proof. If the function $L^2 C_t C^t$ is a function of the point x only on M^n, then the function $C_\alpha C^\alpha$ is constant on I_x. From the covariant differentiation of $C_\alpha C^\alpha$ we get $2\varrho C_\beta = 0$. If ϱ is not equal to zero at a point u of I_x, then we have $\varrho \neq 0$ in a neighbourhood U of u. Thus we have $C_\alpha = 0$ in U. Accordingly, by Theorem 3, we have $D_\beta C_\alpha = \varrho g_{\alpha\beta} = 0$. From this, we have $\varrho = 0$ in U, this is a contradiction. So we have $\varrho = 0$ on I_x. Thus, by Theorem 4, M^n is a Riemannian space.

Again, we shall consider the case of all the indicatrices are conformally flat. If we denote the function $C_t C^t$ by C, then the v-covariant differentiation of C gives us

(3.5) $$C|_i = 2\varrho C_i - 2L^{-1} C l_i.$$

Moreover, v-covariantly differentiating (3.5) and using (1.1c) and (1.1d), we get

(3.6) $$C|_i|_j = 2\varrho|_j C_i + 2(\varrho^2 - L^{-2} C) h_{ij} + 6L^{-2} C l_i l_j - 2L^{-1} \varrho C_i l_j - 6L^{-1} \varrho l_i C_j.$$

From the Ricci's identity for the function C: $C|_i|_j - C|_j|_i = 0$, we have

(3.7) $$\varrho|_j C_i - \varrho|_i C_j + 2L^{-1} \varrho C_i l_j - 2L^{-1} \varrho l_i C_j = 0.$$

Contracting (3.7) by C^i, we have that $C\varrho|_j - \varrho|_t C^t C_j + 2L^{-1} \varrho C l_j = 0$. Therefore if the function C is nonzero, we get

(3.8) $$\varrho|_j = C^{-1} \varrho|_t C^t C_j - 2L^{-1} \varrho l_j.$$

Next, v-covariantly differentiating (3.3) and using (1.1c), (1.1d) and (3.8), we have

(3.9) $$C_i|_j|_k = C^{-1} \varrho|_t C^t h_{ij} C_k - L^{-2} h_{jk} C_i - L^{-2} h_{ik} C_j -$$
$$-2L^{-1} \varrho (h_{ij} l_k + h_{jk} l_i + h_{ki} l_j) + 2L^{-2} (C_i l_j l_k + C_j l_k l_i + C_k l_i l_j).$$

We shall consider the Ricci's identity for the vector C_i: $C_i|_j|_k - C_i|_k|_j = -C_t S_i{}^t{}_{jk}$, then from (3.9) we obtain that

(3.10) $$-C_t S_i{}^t{}_{jk} = (C^{-1} \varrho|_t C^t + L^{-2})(h_{ij} C_k - h_{ik} C_j).$$

Substituting (2.5) in (3.10), we have

(3.11) $$h_{ij} C_{tk} C^t - h_{ik} C_{tj} C^t + C_{ij} C_k - C_{ik} C_j =$$
$$= (C^{-1} \varrho|_t C^t + L^{-2})(h_{ij} C_k - h_{ik} C_j).$$

Contracting (3.11) by g^{ij} and using (1.1b), we get

(3.12) $$C_{tk} C^t = (n-2)(C^{-1} \varrho|_t C^t + L^{-2}) C_k/(n-3) - S C_k/2(n-2)(n-3).$$

Substituting (3.12) in (3.11) and then contracting by C^k,

$$C C_{ij} = \big(S/2(n-2)(n-3) - (C^{-1} \varrho|_t C^t + L^{-2})/(n-3)\big) C h_{ij} +$$
$$+ \big((n-1)(C^{-1} \varrho|_t C^t + L^{-2})/(n-3) - S/(n-2)(n-3)\big) C_i C_j.$$

Therefore we get

Theorem 7. *Let M^n be an $n(\geqq 4)$-dimensional non Riemannian Finsler space. If all the indicatrices are conformally flat and the vector C_α is conformal Killing, then $C_{ij} = \alpha h_{ij} + \beta C_i C_j$, where*

$$\alpha = S/2(n-2)(n-3) - (C^{-1} \varrho|_t C^t + L^{-2})/(n-3),$$
$$\beta = C^{-1} \big\{(n-1)(C^{-1} \varrho|_t C^t + L^{-2})/(n-3) - S/(n-2)(n-3)\big\}.$$

Moreover substituting $C_{ij}=\alpha h_{ij}+\beta C_i C_j$ in (2.8), we get

Corollary 8. *For the functions* α *and* β *in Theorem 7, we have that* $\alpha|_i = C^{-1}\alpha|_t C^t C_i - 2L^{-1}\alpha l_i$ and $\beta|_i = C^{-1}\beta|_t C^t C_i$.

Now we assume that the Finsler space M^n is non Riemannian $S3$-like Finsler space satisfying the T-condition. From the definition of the $S3$-like Finsler space whose v-curvature tensor is written in the form: $S_{ijkl}=S(h_{il}h_{jk}-h_{ik}h_{jl})/(n-1)(n-2)$ and Theorem 7, we have

$$S(h_{il}h_{jk}-h_{ik}h_{jl})/(n-1)(n-2) = \tag{3.13}$$

$$= 2\alpha(h_{il}h_{jk}-h_{ik}h_{jl})+\beta(h_{il}C_jC_k+h_{jk}C_iC_l-h_{ik}C_jC_l-h_{jl}C_iC_k).$$

We get easily that $\beta=0$ by (3.13). Therefore we have that $S=L^{-2}(n-1)(n-2)$. This equation and (1.10) give us that

Theorem 9. *Let M^n be a $S3$-like non Riemannian Finsler space satisfying the T-condition. Then the curvature tensor of the indicatrix I_x $(x\in M^n)$ vanishes identically.*

§ 4. *C-reducible Finsler spaces*

In this section we shall consider a C-reducible Finsler space defined by M. Matsumoto.

Definition. A Finsler space M^n is called C-reducible if the $h(hv)$-torsion tensor C_{ijk} is written in the form:

$$C_{ijk} = (h_{ij}C_k+h_{jk}C_i+h_{ki}C_j)/(n+1). \tag{4.1}$$

Substituting (4.1) in $S_{ijkl}=C_{ilt}C_j{}^t{}_k-C_{ikt}C_j{}^t{}_l$, we have that

$$S_{ijkl} = h_{il}C_{jk}+h_{jk}C_{il}-h_{ik}C_{jl}-h_{jl}C_{ik}, \tag{4.2}$$

where $C_{ij}=(Ch_{ij}/2+C_iC_j)/(n+1)^2$. Therefore we get

Theorem 10. *Let M^n be an $n(\geq 4)$-dimensional C-reducible Finsler space. Then the indicatrix I_x $(x\in M^n)$ is conformally flat.*

Next we shall consider the v-covariant differentiation of C_i. From the definition of the C-reducible Finsler space, it is clear that

$$C_i|_j = \varrho h_{ij}-L^{-1}C_il_j-L^{-1}C_jl_i. \tag{4.3}$$

Therefore we have the following corollaries from Theorem 3, Theorem 4 and Corollary 6.

Corollary 11. *Let M^n be an $(n\geq 3)$-dimensional C-reducible Finsler space. Then the vector C_α is conformal Killing.*

Corollary 12. *Let M^n be an $(n \geqq 3)$-dimensional C-reducible Finsler space and assume that the indicatrices are all closed. If the function $C^t{}_{|t}$ is of constant sign then M^n is a Riemannian space.*

Corollary 13. *Let M^n be an $n(\geqq 3)$-dimensional C-reducible Finsler space and assume that all the indicatrices are closed. If the function $L^2 C_t C^t$ is the function of the point x only, then M^n is a Riemannian space.*

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