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On ideals and extensions of near-rings

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Abstract. Given a chain of ideals $J \triangleleft I \triangleleft N$ in a near-ring N, we consider necessary and sufficient conditions on J, I, N, N/I and I/J respectively to ensure that $J \triangleleft N$.

§1. Introduction

Near-rings considered will be right near-rings; the variety of all near-rings will be denoted by $\mathcal{V}$ and the subvariety of 0-symmetric near-rings will be denoted by $\mathcal{V}_0$. To facilitate discussions, the conditions mentioned in the Abstract will be formulated in terms of a subclass $\mathcal{M}$ of $\mathcal{V}$.

The class $\mathcal{M}$ satisfies condition

(F) If $J \triangleleft I \triangleleft N$ and $I/J \in \mathcal{M}$, then $J \triangleleft N$
(G) If $J \triangleleft I \triangleleft N$ and $J \in \mathcal{M}$, then $J \triangleleft N$
(H) If $J \triangleleft I \triangleleft N$ and $I \in \mathcal{M}$, then $J \triangleleft N$
(K) If $J \triangleleft I \triangleleft N$ and $N \in \mathcal{M}$, then $J \triangleleft N$
(L) If $J \triangleleft I \triangleleft N$ and $N/I \in \mathcal{M}$, then $J \triangleleft N$.

It is our purpose here to describe the near-rings in $\mathcal{M}$ for each of the above five cases. For rings, this has been done, see [8] or Sands [3]. For the purpose of comparison, we recall:

In the variety of rings, a subclass $\mathcal{M}$ satisfies condition:

(F) if and only if the rings in $\mathcal{M}$ are quasi semiprime, i.e. if $R \in \mathcal{M}$ and $xR = 0$ or $Rx = 0$ ($x \in R$), then $x = 0$ (or equivantly, R has zero middle annihilator, i.e. $RxR = 0$ ($x \in R$) implies $x = 0$).
(G) if and only if $R^2 = R$ for all $R \in \mathcal{M}$.
(H) if and only if every ideal I of $R \in \mathcal{M}$ is invariant under all double homothetisms of R (cf Rédei [2]).
(K) if and only if for all $R \in \mathcal{M}$ and for all $a \in R$, $(a) = (a)^2 + \mathbf{Z}a$ where (a) is the ideal in R generated by a and $\mathbf{Z}$ is the integers.

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(L) if and only if $\mathcal{M} = \{0\}$.

If N is a near-ring, then N^+ will denote the underlying group. For nearrings N_i, $i = 1, 2, \ldots, k$ and subsets $U_i \subseteq N_i$, $(U_1, U_2, \ldots, U_k)$ denotes the subset $\{(u_1, u_2, \ldots, u_k) \mid u_i \in U_i\}$ of $N_1 \times N_2 \times \ldots \times N_k$.

§2. On condition (F)

In the variety of 0-symmetric near-rings, this problem has been settled in [9]: A class of near-rings $\mathcal{M}$ in $\mathcal{V}_0$ satisfies condition (F) if and only if every near-ring in $\mathcal{M}$ is quasi semi-equiprime. A near-ring N is *quasi semi-equiprime* if $xN = 0$ $(x \in N)$ implies $x = 0$ and whenever $\theta : I \to N$ is a surjective homomorphism with $I \triangleleft A$ then $x - y \in \ker\theta\,(x, y \in I)$, implies $ax - ay \in \ker\theta$ for all $a \in A$.

In the variety of all near-rings, a complete description is still oustanding. The construction in [7] shows that $\mathcal{M}$ does not contain any non zero constant near-rings. In fact, it is conjectured that $\mathcal{M} = \{0\}$; the strongest motivation for this coming from the example in [6] which shows that any class which contains the two element field cannot satisfy condition (F).

§3. On condition (G)

This case can quickly be disposed of using a construction given in [5] which resembles one given by Betsch and Kaarli [1]. Let N be a near-ring and let K be the near-ring with $K^+ = N^+ \oplus N^+ \oplus N^+$ and with multiplication

$$(a,b,c)(x,y,z) = \begin{cases} (b,0,cz) & \text{if } y \text{ and } z \text{ are non-zero} \\ (0,0,cz) & \text{otherwise.} \end{cases}$$

Apart from verifying the associativity of the multiplication as well as the right distributivity over the addition, it is straightforward to see that $N \cong (0,0,N) \triangleleft (N,0,N) \triangleleft K$, $(0,0,N) \triangleleft K$ if and only if $N = 0$ and K is 0-symmetric if and only if N is 0-symmetric.

Theorem 3.1. *In either one of $\mathcal{V}$ or $\mathcal{V}_0$, a subclass $\mathcal{M}$ satisfies condition (G) if and only if $\mathcal{M} = \{0\}$.*

Proof. Suppose $\mathcal{V}$ satisfies condition G and let $N \in \mathcal{V}$. Then $N \cong (0,0,N) \triangleleft (N,0,N) \triangleleft K$; hence $(0,0,N) \triangleleft K$, which yields $N = 0$.

§4. On condition (L)

As in the variety of rings, a subclass $\mathcal{M}$ of $\mathcal{V}$ or $\mathcal{V}_0$ satisfies condition (L) if and only if $\mathcal{M} = \{0\}$. To verify this for near-rings, we need two constructions:

4.1 Let N be a near-ring and let G be the group $G = N^+ \oplus U$ where U is any non-zero group. As is well-known, N can be identified with a subnear-ring of $M(G) = \{f \mid f : G \to G \text{ a function}\}$ via $\theta : N \to M(G)$, $\theta(n) = \theta_n : G \to G,\ \ \theta_n(g) = \begin{cases} ng & \text{if } g \in N \\ n & \text{if } g \in G \setminus N. \end{cases}$
Let K_1 be the near-ring with $K_1^+ = N^+ \oplus M(G)^+$ and multiplication

$$(n, g)(m, h) = (nm, nh).$$

Let $X = \{f \in M(G) \mid f(U) \subseteq U\}$. Then $(0, X) \triangleleft (0, M(G)) \triangleleft K_1$ and $K_1/(0, M(G)) \cong N$. Note that $(0, X)$ is a right ideal in K_1. It is a left ideal of K_1 if and only if N is constant. Indeed, $(0, X)$ is a left ideal of K_1 if and only if $n(g + h) - ng \in X$ for all $n \in N$, $g \in M(G)$ and $h \in X$. Let g and h be the functions defined by

$$g(x) = \begin{cases} x & \text{if } x \in N \\ 0 & \text{if } x \in G \setminus N \end{cases} \quad \text{and} \quad h(x) = \begin{cases} 0 & \text{if } x \in N \\ x & \text{if } x \in G \setminus N. \end{cases}$$

Clearly $h \in X$; thus if $(0, X) \triangleleft K_1$, then $n(g + h) - ng \in X$. Thus, for $0 \neq u \in U$, $\quad n(g(u) + h(u)) - ng(u) \in U$,

$$\text{i.e.} \quad n(u) - n(0) \in U$$
$$\text{i.e.} \quad n - n0 \in U \cap N = \{0\}; \text{ hence } n = n0.$$

Conversely, if N is constant, then $(0, X) \triangleleft K_1$. If N is 0-symmetric, and we replace $M(G)$ above with $M_0(G)$, then everything above stays valid except in this case $(0, X) \triangleleft K_1$ if and only if $N = 0$.

4.2 Let N be a near-ring and let G be any group which properly contains N^+. We regard N as a subnear-ring of $M(G)$. Let K_2 be the near-ring with $K_2^+ = N^+ \oplus M(G)^+$ and multiplication $(n, f)(m, g) = (nm, fm)$. Then $(0, M_0(G)) \triangleleft (0, M(G)) \triangleleft K_2$ and $K_2/(0, M(G)) \cong N$. Furthermore, $(0, M_0(G))$ is a left ideal in K_2, but it is an ideal if and only if N is 0-symmetric. Indeed, if it is a right ideal, then $(0, 1_G)(n, 0) \in (0, M_0(G))$ which gives $n0 = 0$. The converse is obvious.

Theorem 4.3. *In either one of $\mathcal{V}$ or $\mathcal{V}_0$, a subclass $\mathcal{M}$ satisfies condition (L) if and only if $\mathcal{M} = \{0\}$.*

Proof. Firstly, if $\mathcal{M}$ is a subclass in $\mathcal{V}_0$ which satisfies condition (L) and $N \in \mathcal{M}$, we have from the construction in 4.1 that $N = 0$. If $\mathcal{M}$ is a subclass in $\mathcal{V}$ which satisfies condition (L) and $N \in \mathcal{M}$, the construction in 4.1 shows that N is constant and the construction in 4.2 shows that N is 0-symmetric. Hence $N = 0$.

§5. On condition (K)

Principal ideals in a near-ring, contrary to the ring case, have no nice finite description. This seems to be the most serious obstacle in describing the near-rings in a class $\mathcal{M}$ which satisfies condition (K). For a near-ring N and $a \in S \subseteq N$, S a subnear-ring of N, the ideal in S generated by a will be denoted by $\langle a, S\rangle$.

Theorem 5.1. *Let $\mathcal{M}$ be a class of near-rings. Then $\mathcal{M}$ satisfies condition (K) if and only if $\langle a, N\rangle = \langle a, \langle a, N\rangle\rangle$ for all $a \in N$, $N \in \mathcal{V}$.*

Proof. Firstly, if $a \in N \in \mathcal{M}$ and $\mathcal{M}$ satisfies condition (K), then $\langle a, \langle a, N\rangle\rangle \triangleleft N$. Hence $\langle a, N\rangle = \langle a, \langle a, N\rangle\rangle$. Conversely, if the condition is satisfied, choose $J \triangleleft I \triangleleft N \in \mathcal{M}$ and let $j \in J,\ n, m \in N$. Then $\langle j, N\rangle \cap J \subseteq \langle j, N\rangle \subseteq I$; hence

$$\langle j, N\rangle = \langle j, \langle j, N\rangle\rangle \subseteq \langle j, N\rangle \cap J \subseteq J.$$

Thus $n + j - n$, jn and $n(m + j) - nm \in J$ which yields $J \triangleleft N$.

§6. On condition (H)

We start with finding the near-ring analogue of the Schreier group extensions or the Everett ring extensions (cf Rédei [2]): Given near-rings A and B determine all near-rings N such that N is an extension of A by B, i.e. $A \triangleleft N$ and $N/A = B$. This problem has earlier been settled for composition rings and 0–symmetric near-rings by Steinegger [4]. Strictly speaking, it involves finding all triples (ζ, N, η) where

$$0 \longrightarrow A \xrightarrow{\zeta} N \xrightarrow{\eta} B \longrightarrow 0$$

is a short exact sequence. Two extensions (ζ, N, η) and (ζ', N', η') of A by B are *equivalent* if there exists an isomorphism $\chi : N \to N'$ such that the diagram commutes. χ is called an *equivalence isomorphism*. In order to simplify notation and discussions, our exposition will not be as rigorous as required above; instead, we identify A with $\zeta(A)$ and $N/\zeta(A)$ with B. In such a case, it means $\chi(a) = a$ for all $a \in A$ and $\chi(n) + A = n + A$ for all $n \in N$. The elements of A and B will always be taken as $A = \{a, b, c, \dots\}$ and $B = \{\alpha, \beta, \gamma, \dots\}$ respectively with 0 denoting the additive identity of both A and B.

Consider a quintuple of functions $(F, [-,-], G, H, \langle -,-\rangle)$ with $F : B \to M(A)$, $[-,-] : B \times B \to A$, $G : A \times B \to M(A)$, $H : B \times B \to M(A)$ and $\langle -,-\rangle : B \times B \to A$ which satisfy the *initial conditions*

$$\begin{aligned}
&F(\alpha) \in M_0(A),\ F(0) = 1_A \\
&[\alpha, 0] = 0 = [0, \alpha] \\
&G(b, \beta) \in M_0(A), G(b, 0)(a) = ab \\
&H(0, \beta) = 0;\ H(\alpha, \beta) \in M_0(A) \\
&\langle 0, \beta\rangle = 0.
\end{aligned}$$

On the cartesian product $A \times B$ define two operations by

$$(a, \alpha) + (b, \beta) = \big(\, a + F(\alpha)(b) + [\alpha, \beta],\ \alpha + \beta \,\big)$$

and

$$(a, \alpha)(b, \beta) = \big(\, G(b, \beta)(a) + H(\alpha, \beta)(b) + \langle \alpha, \beta\rangle,\ \alpha\beta \,\big)\ .$$

We say these two operations are the operations induced by the function quintuple. $A \times B$ together with these two operations is called an *E-sum of A and B* w.r.t. the quintuple $(F, [-,-], G, H, \langle -,-\rangle)$ and is denoted by $A \sharp B$. The two functions $[-,-]$ and $\langle -,-\rangle$ are called *the additive and multiplicative factor systems* of the E-sum respectively.

Although A is (as a near-ring) isomorphic to $(A, 0)$ (via $a \to (a, 0)$), in general $A \sharp B$ may have no particular structure w.r.t. the induced operations. A function quintuple $(F, [-,-], G, H, \langle -,-\rangle)$ is called an *amicable system for A w.r.t.* B if it satisfies the following conditions for all $a, b, c \in A$ and $\alpha, \beta, \gamma \in B$:

(E1) $F(\alpha) \in \mathrm{End}\,(A^+)$

(E2) $[\alpha, \beta] + F(\alpha + \beta)(c) = F(\alpha)\,\big(\, F(\beta)(c) \,\big) + [\alpha, \beta]$

(E3) $[\alpha, \beta] + [\alpha + \beta, \gamma] = F(\alpha)\,\big(\, [\alpha, \beta] \,\big) + [\alpha, \beta + \gamma]$

(E4) $G(c, \gamma) \in \mathrm{End}\,(A^+)$

(E5) $G(c, \gamma)\,\big(\, F(\alpha)(b) \,\big) + G(c, \gamma)\,\big(\, [\alpha, \beta] \,\big) + H(\alpha + \beta, \gamma)(c) + \langle \alpha + \beta, \gamma\rangle$
$= H(\alpha, \gamma)(c) + \langle \alpha, \gamma\rangle + F(\alpha\gamma)\,\big(\, G(c, \gamma)(b) \,\big) + F(\alpha\gamma)\,\big(\, H(\beta, \gamma)(c) \,\big)$
$+ F(\alpha\gamma)\,\big(\, \langle \beta, \gamma\rangle \,\big) + [\alpha\gamma, \beta\gamma]$

(E6) $G(c,\gamma)\left(\,H(\alpha,\beta)(b)\,\right) + G(c,\gamma)\left(\,\langle\alpha,\beta\rangle\,\right) + H(\alpha\beta,\gamma)(c) + \langle\alpha\beta,\gamma\rangle$
$= H(\alpha,\beta\gamma)\left(\,G(c,\gamma)(b) + H(\beta,\gamma)(c) + \langle\beta,\gamma\rangle\,\right) + \langle\alpha,\beta\gamma\rangle$

(E7) $G(c,\gamma)\circ G(b,\beta) = G\left(\,G(c,\gamma)(b) + H(\beta,\gamma)(c) + \langle\beta,\gamma\rangle, \beta\gamma\,\right)$.

Theorem 6.1. *Let $A\sharp B$ be an E-sum of A and B w.r.t. a function quintuple $(F,[-,-],G,H,\langle-,-\rangle)$. Then $A\sharp B$ is a near-ring if and only if the function quintuple is an amicable system for A w.r.t. B. In such a case, $A\sharp B$ is an extension of A by B.*

Proof. Assume $A\sharp B$ is a near-ring. Since the addition is associative, it follows that

$$\begin{aligned} &F(\alpha)(\beta) + [\alpha,\beta] + F(\alpha+\beta)(c) + [\alpha+\beta,\gamma] \\ &= F(\alpha)\left(\,b + F(\beta)(c) + [\beta,\gamma]\,\right) + [\alpha,\beta+\gamma] \end{aligned} \tag{1}$$

From this equality and the initial conditions, (E1), (E2) and (E3) are obtained by putting $\beta=\gamma=0$, $b=\gamma=0$ and $b=c=0$ respectively. The right distributivity of the multiplication over the addition, using (E1), gives

$$\begin{aligned} &G(c,\gamma)\left(\,a + F(\alpha)(b) + [\alpha,\beta]\,\right) + H(\alpha+\beta,\gamma)(c) + \langle\alpha+\beta,\gamma\rangle \\ &= G(c,\gamma)(a) + H(\alpha,\gamma)(c) + \langle\alpha,\gamma\rangle + F(\alpha\gamma)\left(\,G(c,\gamma)(b)\,\right) \\ &+ F(\alpha\gamma)\left(\,H(\beta,\gamma)(c)\,\right) + F(\alpha\gamma)(\langle\beta,\gamma\rangle) + [\alpha\gamma,\beta\gamma] \end{aligned} \tag{2}$$

Substituting $\alpha=\beta=0$ yields (E4). Then using this in (2) gives (E5). Using (E4), the associativity of the multiplication gives

$$\begin{aligned} &G(c,\gamma)\left(\,G(b,\beta)(a)\,\right) + G(c,\gamma)\left(\,H(\alpha,\beta)(b)\,\right) + G(c,\gamma)\left(\,\langle\alpha,\beta\rangle\,\right) \\ &+ H(\alpha\beta,\gamma)(c) + \langle\alpha\beta,\gamma\rangle \\ &= G\left(\,G(c,\gamma)(b) + H(\beta,\gamma)(c) + \langle\beta,\gamma\rangle, \beta\gamma\,\right)(a) \\ &+ H(\alpha,\beta\gamma)\left(\,G(c,\gamma)(b) + H(\beta,\gamma)(c) + \langle\beta,\gamma\rangle\,\right) + \langle\alpha,\beta\gamma\rangle \end{aligned} \tag{3}$$

Substituting $a=0$ and $\alpha=0$ in this equality, (E6) and (E7) respectively are obtained.

Conversely, if the function quintuple is an amicable system, we verify that $A\sharp B$ is a near-ring. The associativity of the addition will follow if the equality in (1) holds. Using (E1), (E3) and then (E2), the right hand side of (1) becomes:

$$\begin{aligned} &F(\alpha)\left(\,b + F(\beta)(c) + [\beta,\gamma]\,\right) + [\alpha,\beta+\gamma] \\ &\quad = F(\alpha)(b) + F(\alpha)\left(\,F(\beta)(c)\,\right) + F(\alpha)\left(\,[\beta,\gamma]\,\right) + [\alpha,\beta+\gamma] \\ &\quad = F(\alpha)(b) + F(\alpha)\left(\,F(\beta)(c)\,\right) + [\alpha,\beta] + [\alpha+\beta,\gamma] \\ &\quad = F(\alpha)(b) + [\alpha,\beta] + F(\alpha+\beta)(c) + [\alpha+\beta,\gamma]\,. \end{aligned}$$

It can be verified that $(0,0)$ is the additive identity and every element (a,α) has an additive inverse $-(a,\alpha)=\big(-[-\alpha,\alpha]-F(-\alpha)(a),-\alpha\big)$. Hence $A\sharp B$ is a group. For the right distributivity, we need the equality in (2). Using E(4), the left hand side becomes

$$G(c,\gamma)(a)+G(c,\gamma)\big(F(\alpha)(b)\big)+G(c,\gamma)\big([\alpha,\beta]\big)+H(\alpha+\beta,\gamma)(c)+\langle\alpha+\beta,\gamma\rangle$$

which equals the right hand side in view of (E5).

Finally, for the associativity, we require the equality in (3). Using (E7) and (E6), the right hand side becomes

$$\begin{aligned}&G(c,\gamma)\big(G(b,\beta)(a)\big)+G(c,\gamma)(H(\alpha,\beta)(b))+G(c,\gamma)\big(\langle\alpha,\beta\rangle\big)\\&+H(\alpha\beta,\gamma)(c)+\langle\alpha\beta,\gamma\rangle=G\big(G(c,\gamma)(b)+H(\beta,\gamma)(c)+\langle\beta,\gamma\rangle,\beta\gamma\big)(a)\\&+H(\alpha,\beta\gamma)\big(G(c,\gamma)(b)+H(\beta,\gamma)(c)+\langle\beta,\gamma\rangle\big)+\langle\alpha,\beta\gamma\rangle.\end{aligned}$$

Thus $A\sharp B$ is a near-ring. If we identify A with $(A,0)$ and B with $\{(0,\alpha)+(A,0)\,|\,\alpha\in B\}$, we have $A\triangleleft A\sharp B$ and $A\sharp B/A=B$.

Remark. Conditions (E1), (E2) and (E3), which are equivalent to $A\sharp B$ being a group under the induced addition, implies $F(\alpha)\in\operatorname{Aut}(A^+)$ for all $\alpha\in B$. Indeed, by (E1) we only have to verify that $F(\alpha)$ is bijective. If $F(\alpha)(a)=F(\alpha)(b)$, then

$$(0,\alpha)+(a,0)=\big(F(\alpha)(a),\alpha\big)=\big(F(\alpha)(b),\alpha\big)=(0,\alpha)+(b,0).$$

Hence $(a,0)=(b,0)$ which yields the injectivity. Substituting $\beta=0$ in (E2) gives $F(\alpha)(c)=F(\alpha)\big(F(0)(c)\big)$; hence $c=F(0)(c)$ which yields $F(0)=1_A$. Using (E3) with $\beta=-\alpha$ and $\gamma=\alpha$, gives $[\alpha,-\alpha]=F(\alpha)\big([-\alpha,\alpha]\big)$. Thus, for any $c\in A$, using (E2) with $\beta=-\alpha$ and (E1) yield $F(\alpha)\big([-\alpha,\alpha]\big)+c=F(\alpha)\big(F(-\alpha)(c)\big)+F(\alpha)\big([-\alpha,\alpha]\big)$, i.e.

$$c=F(\alpha)\big(-[-\alpha,\alpha]+F(-\alpha)(c)+[-\alpha,\alpha]\big)$$

which shows that $F(\alpha)$ is surjective.
All extensions of A by B are, up to equivalence, an E-sum of A and B for a suitable amicable system. This is our next result.

Theorem 6.2. *Let A and B be near-rings and let N be an extension of A by B. Then N is equivalent to an E-sum $A\sharp B$ for some amicable system $(F,[-,-],G,H,\langle-,-\rangle)$.*

Proof. Each $\alpha\in B=N/A$ is a subset; let f be a choice function with $f(\alpha)\in\alpha$ and $f(0)=0$. Every element $n\in N$ can uniquely be expressed as $n=a+f(\alpha)$ for some $a\in A$, $\alpha\in B$. Define a function

$\phi : N \to A \times B$ by $\phi\big(\ a + f(\alpha)\ \big) = (a, \alpha)$. Then ϕ is an injection. Define a quintuple of functions by:

$$F : B \to M(A), \quad F(\alpha)(a) = f(\alpha) + a - f(\alpha)$$
$$[-,-] : B \times B \to A, \quad [\alpha, \beta] = f(\alpha) + f(\beta) - f(\alpha + \beta)$$
$$G : A \times B \to A, \quad G(b, \beta)(a) = a\ \big(\ b + f(\beta)\ \big)$$
$$H : B \times B \to M(A) \quad \text{by} \quad H(\alpha, \beta)(a) = f(\alpha)[a + f(\beta)] - f(\alpha)f(\beta)$$
$$\langle -,- \rangle : B \times B \to A \quad \text{by} \quad \langle \alpha, \beta \rangle = f(\alpha)f(\beta) - f(\alpha\beta).$$

These functions are all well-defined; for example, we verify it for H : Since $f(\alpha) \in \alpha$ and $f(\beta) \in \beta$, $f(\alpha) = n_1 + a_1$ and $f(\beta) = n_2 + a_2$ for suitable $n_1, n_2 \in N$, $a_1, a_2 \in A$. Then

$$f(\alpha)\ \big[\ a+f(\beta)\ \big]\ -f(\alpha)f(\beta) = (n_1+a_2)\ \big[\ a+(n_1+a_2)\ \big]\ -(n_1+a_1)(n_2+a_2)$$

which is in A since $A \triangleleft N$. The quintuple $(F,\ [-,-],\ G,\ H,\ \langle -,- \rangle)$ satisfies the initial conditions since $f(0) = 0$. In addition, we will show that they form an amicable system for A w.r.t. B. But this will follow if we can show that $A\sharp B$ is a near-ring respect to the addition and multiplication induced by this function quintuple. To this effect, it is sufficient to show that ϕ preserves addition and multiplication. Firstly we note that

$$\begin{aligned}\big(\ a + f(\alpha)\ \big) + \big(\ b + f(\beta)\ \big) &= a + f(\alpha) + b - f(\alpha) + f(\alpha) + f(\beta) \\ &\quad - f(\alpha + \beta) + f(\alpha + \beta) = a + F(\alpha)(b) + [\alpha, \beta] + f(\alpha + \beta).\end{aligned}$$

The first three terms are in A; hence it is the unique expression of $\big(\ a + f(\alpha)\ \big) + \big(\ b + f(\beta)\ \big) \in N$ in the form $c + f(\gamma)$. Thus

$$\begin{aligned}\phi\Big(\big(\ a + f(a)\ \big) + \big(\ b + f(\beta)\ \big)\Big) &= \big(\ a + F(\alpha)(b) + [\alpha, \beta],\ \alpha + \beta\ \big) \\ &= (a, \alpha) + (b, \beta).\end{aligned}$$

Likewise,

$$\begin{aligned}&\big(\ a + f(\alpha)\ \big)\big(\ b + f(\beta)\ \big) \\ &= a\ \big(\ b + f(\beta)\ \big) + f(\alpha)\ \big(\ b + f(\beta)\ \big) - f(\alpha)f(\beta) \\ &\quad + f(\alpha)f(\beta) - f(\alpha\beta) + f(\alpha\beta) \\ &= G(b, \beta)(a) + H(\alpha, \beta)(b) + \langle \alpha, \beta \rangle + f(\alpha\beta)\end{aligned}$$

and the first three are terms in A. Hence

$$\begin{aligned}\phi\Big(\big(\ a + f(\alpha)\ \big)\big(\ b + f(\beta)\ \big)\Big) &= \big(\ G(b, \beta)(a) + H(\alpha, \beta)(b) + \langle \alpha, \beta \rangle,\ \alpha\beta\ \big) \\ &= (a, \alpha)(b, \beta).\end{aligned}$$

Hence ϕ is a near-ring isomorphism. In fact, it is an equivalence isomorphism: If $a \in A$, then $\phi(a) = \phi(a + 0) = (a, 0)$. As usual, we identify

$\{n + A \,|\, n \in N\}$ with $A\sharp B/(A,0) = \big\{ \ (0,\alpha) + (A,0) \,|\, \alpha \in B \ \big\}$ via $n = a + f(\alpha)$ for some unique $a \in A$, $\alpha \in B$. Then

$$\phi(n) + (A,0) = (a,\alpha) + (A,0) = (0,\alpha) + (A,0) = n + A.$$

Remark. For any two near-rings A and B there always exists at least one E-sum $A\sharp B$ with an amicable system of functions, namely $F(\alpha) = 1_A$, $[\alpha,\beta] = 0 = \langle\alpha,\beta\rangle$, $G(c,\gamma)(a) = ac$ and $H(\alpha,\beta) = 0$. This is nothing but the direct sum $A \oplus B$ of the near-rings A and B.

An amicable system $(F,\ [-,-],\ G,\ H,\ \langle -,-\rangle)$ for A w.r.t. B is called a *factor-free amicable system* if $[\alpha,\beta] = 0 = \langle\alpha,\beta\rangle$. In such a case, it will be denoted by (F,G,H) and the initial conditions and the conditions (E1) to (E7) simplify to:

$F : B \to \mathrm{Aut}\,(A^+)$ is a group homomorphism (i.e. $F(\alpha+\beta) = F(\alpha) \circ F(\beta)$).

$G : A \times B \to \mathrm{End}\,(A^+)$ and

$H : B \times B \to M_0(A)$ are functions with $H(0,\alpha) = 0$ and

$$(F1)\ G(c,\gamma)\ \big(\ F(\alpha)(b)\ \big) + H(\alpha+\beta,\gamma)(c) = H(\alpha,\gamma)(c) + F(\alpha\gamma)\ \big(\ G(c,\gamma)(b)\ \big) + F(\alpha\gamma)\ \big(\ H(\beta,\gamma)(c)\ \big)$$

$$(F2)\ G(c,\gamma)\ \big(\ H(\alpha,\beta)(b)\ \big) + H(\alpha\beta,\gamma)(c) = H(\alpha,\beta\gamma)\ \big(\ G(c,\gamma)(b) + H(\beta,\gamma)(c)\ \big)$$

$$(F3)\ G(c,\gamma) \circ G(b,\beta) = G\ \big(\ G(c,\gamma)(b) + H(\beta,\gamma)(c),\ \beta\gamma\ \big)\ .$$

((F1), (F2) and (F3) follows from (E5), (E6) and (E7) respectively.) A triple (f,g,h), where $f,g,h \in M_0(A)$, is called a *triple homothetism of* A if $f = F(\alpha)$, $g = G(b,\beta)$ and $h = H(\beta,\alpha)$ for some $b \in A$, $\alpha,\beta \in B$ where (F,G,H) is a factorfree amicable system for A with respect to some B. If $I \triangleleft A$, then I is *invariant under the triple homothetism* (f,g,h) if $f(I) = I$, $g(I) \subseteq I$ and $h(I) \subseteq I$.

Theorem 6.3. *Let $\mathcal{M}$ be a class of near-rings. Then $\mathcal{M}$ satisfies condition (H) if and only every ideal $I \triangleleft A$ for $A \in \mathcal{M}$ is invariant under every triple homothetism of A.*

Proof. Let $I \triangleleft A \in \mathcal{M}$, $\mathcal{M}$ satisfies condition (H), and let (f,g,h) be a triple homothetism of A. By definition, there is a near-ring B and a factorfree amicable system (F,G,H) for A w.r.t. B such that $f = F(\alpha_0)$, $g = G(b_0,\beta_0)$ and $h = H(\beta_0,\alpha_0)$. The E-sum $A\sharp B$ is a near-ring, w.r.t. the operations induced by F, G and H, and $(I,0) \triangleleft (A,0) \triangleleft A\sharp B$. By condition (H), $(I,0) \triangleleft A\sharp B$; hence $(0,\alpha) + (i,0) - (0,\alpha) \in (I,0)$. This means $F(\alpha)(i) \in I$ and in particular $f(i) = F(\alpha_0)(i) \in I$. Thus $f(I) \subseteq I$.

For the reverse inclusion, since $F(\alpha)(i) \in I$ for all α, we have $i = F(\alpha_0)(F(-\alpha_0)(i)) \in F(\alpha_0)(I) = f(I)$; hence $f(I) = I$. Furthermore, $(i,0)(b,\beta) \in (I,0)$; hence $G(b,\beta)(i) \in I$ and in particular, $g(i) = G(b_0,\beta_0)(i) \in I$. Thus $g(I) \subseteq I$. Lastly, from $(0,\beta)[(0,\alpha)+(i,0)] - (0,\beta)(0,\alpha) \in (I,0)$ we have $H(\beta,\alpha)(F(\alpha)(i)) \in I$. In particular for $\beta = \beta_0$ and $\alpha = \alpha_0$ and since the restriction of $F(\alpha_0)$ to I is an automorphism of I, $h(i) = H(\beta_0,\alpha_0)(F(\alpha_0)(j)) \in I$ for a suitable $j \in I$. Thus $h(I) \subseteq I$.

Conversely, suppose the ideals of $A \in \mathcal{M}$ are invariant under triple homothetism of A. Consider $I \triangleleft A \triangleleft B$. Define functions $F : B \to \mathrm{Aut}\,(A^+)$, $G : A \times B \to \mathrm{End}\,(A^+)$ and $H : B \times B \to M_0(A)$ by $F(\alpha)(a) = \alpha + a - \alpha$, $G(a,\beta)(c) = c(a+\beta)$ and $H(\alpha,\beta)(a) = \alpha(a+\beta) - \alpha\beta$. Then (F,G,H) constitutes a factorfree amicable system for A w.r.t. B. Since I is invariant under triple homothetisms and $(F(\alpha),\ G(b,\beta),\ H(\beta,\alpha))$ is a triple homothetism for all $b \in A$, $\alpha,\beta \in B$, it follows that $I \triangleleft B$.

In conclusion we may mention that in the ring case, it is possible to express a double homothetism (in this case, $F(\alpha) = 1_A$ for all α and is thus omitted in the triple (f,g,h)) only in terms of A and the conditions (E1) to (E7) simplify considerably.

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