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Boundedness and stability of solutions of a certain nonlinear differential equation of the second order

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In this paper we shall discuss the boundedness and stability of solutions of the second order differential equation

$$(1)\quad a(t)x'' + b(t)f(x,x') + g(t,x,x') + (1+c(t))h(t,x)l(x') = e(t,x,x'),$$

which is equivalent to the system

$$(1')\quad \begin{aligned} x' &= y, \\ y' &= \frac{1}{a(t)}(e(t,x,y) - b(t)f(x,y) - g(t,x,y) - (1+c(t))h(t,x)l(y)). \end{aligned}$$

In the first part of this paper there are introduced some sufficient conditions for a solution $(x(t), y(t))$ of the system (1') to be defined and bounded. Further, in this part there are given some results concerning the uniform boundedness of solutions and the uniform asymptotic stability of the trivial solution of the system (1'). These results generalize or complete some results of [4].

In the second part some necessary conditions are given for a solution $(x(t), y(t))$ of the system (1') to be defined and bounded. Our results of this part generalize some results of [1].

Definitions and Propositions.

In this paper we shall use the following definitions and propositions of [2] and [3].

Let $\varphi(t) = \varphi(t; t_0, x_0)$ denote a solution of the system

$$(2)\qquad x' = f(t,x), \quad x \in R_n,\ t \in I = \langle 0, \infty),\ f(t,x) \in C(I \times R_n)$$

through x_0 at $t = t_0$.

Definition 1 (see [3]). The solutions of (2) are uniformly bounded if for any $\alpha > 0$, there exists $\beta(\alpha) > 0$ such that $|x_0| \leq \alpha$ and $t_0 \geq 0$ imply $|\varphi(t; t_0, x_0)| = \beta(\alpha)$ for $t \geq t_0$.

Proposition 1 (T. Yoshizava [5]). *Let there exist continuous functions $V(t,x)$ and $W_i(x)$, $i=1,2$ in $I\times R_n$ such that the following conditions hold:*

1. $0 < W_1(x) \le V(t,x) \le W_2(x)$, $W_1(x)\to\infty$, $|x|\to\infty$.
2. $V'(t,x) = \dfrac{\partial V}{\partial t} + \dfrac{\partial V}{\partial x} f(t,x) \le 0$.

Then the solutions of (2) are uniformly bounded for $t \ge 0$.

Definition 2 (see [2]). A function $V(t,x)$ is called uniformly small if there exists a continuous, positive definite function $W(x)$ such that

$$V(t,x) \le W(x)$$

in $I \times R_n$.

Proposition 2 (see [2]). *If there exists a continuous, positive definite function $V(t,x)$ with a negative semidefinite total derivative with respect to t, then the trivial solution of (2) is stable (in the sense of Liapunov).*

Proposition 3 (see [2]). *If there exists a continuous, positive definite and uniformly small function $V(t,x)$ with a negatively definite total derivative with respect to t, then the trivial solution of (2) is uniformly asymptotically stable.*

Let us introduce the following notation:

$R_1 = (-\infty,\infty)$, $R_2 = R_1 \times R_1$, $D_1 = I \times R_1$ and $D_2 = I\times R_2$.

We assume that for functions in (1) the following holds: $a \in C^1(I)$, $a(t) \ne 0$ for $t\in I$; $b\in C(I)$, $c\in C^1(I)$, $f \in C(R_2)$, $h\in C(D_1)$, $\dfrac{\partial h(t,x)}{\partial t} \in C(D_1)$, $e\in C(D_2)$, $g \in C(D_2)$ and $l \in C(R_1)$.

Let us define functions $H(t,x)$ in D_1 and $L(y)$ in R_1 in the following way:

$$H(t,x) = \int_0^x h(t,s)\,ds, \qquad L(y) = \int_0^y \frac{s}{l(s)}\,ds.$$

Further let us introduce the following notation: $\{u(t)\}_+ = \max\{u(t),0\}$ and $\{u(t)\}_- = \max\{-u(t),0\}$. Evidently $u(t) = \{u(t)\}_+ - \{u(t)\}_-$ for any real function u in R_1.

1. The sufficient conditions

In what follows, the following conditions will be required:

(1.1) $a(t) > 0$ for $t\in I$;

(1.2) $1 + c(t) > 0$ for $t \in I$;
(1.3) $l(y) > 0$ for $y \in R_1$;
(1.4) $L(y) \to \infty$ for $|y| \to \infty$;
(1.5) $xh(t,x) > 0$ for $x \neq 0$ and $t \in I$;
(1.6) $yg(t,x,y) \geq 0$ in D_2;
(1.7) $b(t) \geq 0$ for every $t \in I$ and $yf(x,y) \geq 0$ for every $x, y \in R_1$ or $b(t) \leq 0$ for every $t \in I$ and $yf(x,y) \leq 0$ for every $x, y \in R_1$;
(1.8) There exists a positive function $p_1 \in C(R_1)$ such that $|xp_1(x)| \leq |h(t,x)|$ in D_1 and

$$P_1(x) = \int_0^x sp_1(s)\,ds \longrightarrow \infty \qquad \text{for } |x| \to \infty.$$

We have

Theorem 1.1. *Suppose that the assumptions (1.1)–(1.6), (1.8) and the following conditions are satisfied:*

(1.9) *There exists a continuous function φ such that*

$$\frac{\partial H(t,x)}{\partial t} \leq \varphi(t)H(t,x) \text{ in } D_1;$$

(1.10) $a(t)\{\varphi(t)\}_+ - a'(t) \leq 0$ *for every* $t \in I$;
(1.11) *There exists a positive constant a_1 such that $a(t) \leq a_1$ for every $t \in I$;*
(1.12) *There exists a positive constant c_1 such that $c(t) \leq c_1$ for every $t \in I$;*
(1.13) $c'(t) \geq 0$ *for every* $t \in I$;
(1.14) $ye(t,x,y) \leq yb(t)f(x,y)$ *in* D_2.

Then all solutions $(x(t), y(t))$ of (1') are defined and bounded on I.

Proof. Let for some $T > 0$ a solution $(x(t), y(t))$ of (1') be defined on $\langle 0, T)$ and let $|x(t)| + |y(t)| \to \infty$ for $t \to T_-$. Define the function

$$V(t,x,y) = \frac{1}{a(t)}H(t,x) + \frac{1}{1+c(t)}L(y) \text{ in } D_2$$

and differentiate $V(t) = V(t, x(t), y(t))$ with respect to t for any solution of (1'). Then by (1.1)–(1.3), (1.5), (1.6) and (1.9)–(1.14) we get

$$\begin{aligned} V'(t) \leq {} & \frac{a(t)\{\varphi(t)\}_+ - a'(t)}{a^2(t)}H(t,x) + \\ & + \frac{1}{1+c(t)} \cdot \frac{1}{a(t)} \cdot \frac{y}{l(y)}\big(e(t,x,y) - b(t)f(x,y) - g(t,x,y)\big) \leq 0 \end{aligned} \tag{3}$$

for every $t \in \langle 0, T)$.

Integrating (3) from 0 to t, $t \in \langle 0, T)$ we obtain $V(t) \leq V(0) = C_0 < \infty$. From (1.1), (1.2) and (1.5) it follows

$$L(y) \leq V(t) \quad \text{for every} \quad t \in \langle 0, T),$$

and using the above inequality for every $t \in \langle 0, T)$ we get

$$L(y) \leq C_1 < \infty. \qquad (C_1 = (1 + c_1)C_0)$$

The condition (1.4) gives that $y(t)$ is bounded on $\langle 0, T)$, thus $x(t)$ too is bounded on $\langle 0, T)$. We have a contradiction. This means that all solutions $(x(t), y(t))$ of (1') are defined on I.

Further we will prove that all solutions $(x(t), y(t))$ of (1') are bounded on I. It is evident that $V(t) \leq C_0 < \infty$ for every $t \in I$. Therefore analogously as above $L(y) \leq C_1 < \infty$ and $H(t, x) \leq C_2 < \infty$ $(C_2 = a_1 C_0)$ for every $t \in I$ and $x \in R_1$, too. In the end by (1.4) and (1.8) the solution $(x(t), y(t))$ of (1') is bounded on I. This completes the proof.

Corollary 1.1. *Let the hypotheses of Theorem* 1.1. *hold. Moreover, suppose that the following condition is satisfied:*

(1.15) $f(0,0) = 0$ *and* $e(t, 0, 0) = 0$ *for every* $t \in I$.

Then the trivial solution of (1') *is stable in the sense of Liapunov.*

PROOF. From (1.5), (1.6) and (1.15) it follows that (1') has the trivial solution. By (1.5), (1.8), (1.11) and (1.12) for the same function $V(t, x, y)$ as in the proof of Theorem 1.1. we have

$$0 \leq W(x, y) = \frac{1}{a_1} P_1(x) + \frac{1}{1 + c_1} L(y) \leq V(t, x, y) \text{ in } D_2.$$

This means that the function $V(t, x, y)$ is positive definite. From (3) it follows that the function $V(t, x, y)$ is negative semidefinite. Therefore by Proposition 2 the trivial solution of (1') is stable. This completes the proof.

If $g(t, x, y) = 0$ in D_2, then the next Theorem gives sufficient conditions for the uniform boundedness of the solutions of (1').

Theorem 1.2. *Let* (1.3)–(1.5), (1.7), (1.8) *and* (1.11)–(1.13) *hold. Moreover, suppose that the following conditions are satisfied:*

(1.16) $a'(t) \leq 0$ *for every* $t \in I$;

(1.17) $\dfrac{\partial H(t, x)}{\partial t} \leq 0$ *in* D_1;

(1.18) *There exists a positive constant* a_2 *such that* $a(t) \geq a_2$ *for every* $t \in I$;

(1.19) *There exists a positive constant* c_2 *such that* $c(t) \geq c_2$ *for every* $t \in I$;

(1.20) *There exists a function* $p_2 \in C(R_1)$ *such that* $xp_2(x) > 0$ *for* $x \neq 0$ *and* $|h(t,x)| \leq |p_2(x)|$ *in* D_1;

(1.21) $ye(t,x,y) \leq 0$ *in* D_2.

Then the solutions of (1') *are uniformly bounded for* $t \geq 0$.

PROOF. For any solution $(x(t), y(t))$ of (1') and a positive K_1 we define

$$V(t,x,y) = H(t,x) + \frac{a(t)}{1+c(t)} L(y) + K_1 \text{ in } D_2.$$

By (1.3), (1.5), (1.8), (1.11)–(1.12) and (1.18)–(1.20) we get

$$\begin{aligned} 0 < W_1(x,y) = P_1(x) + \frac{a_2}{1+c_1} L(y) + K_1 \leq V(t,x,y) \leq \\ \leq P_2(x) + \frac{a_1}{1+c_2} L(y) + K_1 = W_2(x,y) \end{aligned}$$

in D_2 (where $P_2(x) = \int\limits_0^x p_2(s)\,ds$). From (1.4) and (1.8) it follows that $W_1(x,y) \to \infty$ for $|x| \to \infty$ and $|y| \to \infty$.

Differentiating the function $V(t) = V(t, x(t), y(t))$ with respect to t, by (1.3), (1.7), (1.13), (1.16)–(1.19) and (1.21) we obtain

$$\begin{aligned} V'(t) = \frac{\partial H(t,x)}{\partial t} + \frac{a'(t)(1+c(t)) - c'(t)a(t)}{(1+c(t))^2} L(y) + \\ + \frac{1}{1+c(t)} \; \frac{y}{l(y)} (e(t,x,y) - b(t)f(x,y)) \leq 0 \text{ in } D_2. \end{aligned} \tag{4}$$

This means that all conditions of Proposition 1 are fulfilled, therefore the solutions of (1') are uniformly bounded for $t \geq 0$. This completes the proof.

Corollary 1.2. *Let the hypotheses of Theorem 1.2. hold with the exception of* (1.7). *Moreover, suppose that the following condition is satisfied:*

(1.22) *There exists a positive constant* b_1 *such that* $b(t) \geq b_1$ *for every* $t \in I$ *and* $yf(x,y) \geq 0$ *for every* $x, y \in R_1$ *or there exists a negative constant* b_2 *such that* $b(t) \leq b_2$ *for every* $t \in I$ *and* $yf(x,y) \leq 0$ *for every* $x, y \in R_1$.

Then the trivial solution of (1') *is uniformly asymptotically stable.*

PROOF. From (1.5), (1.21) and (1.22) it follows that $h(t,0) = 0$, $e(t,x,0) = 0$ and $f(x,0) = 0$ for every $t \in I$ and $x \in R_1$. This means that (1') has the trivial solution. For any solution $(x(t), y(t))$ of (1') we define

$$V(t,x,y) = H(t,x) + \frac{a(t)}{1+c(t)} L(y) \text{ in } D_2.$$

By (1.3), (1.5), (1.8), (1.11), (1.12) and (1.18)–(1.20) we get

$$V_1(x,y) = P_1(x) + \frac{a_2}{1+c_1}L(y) \le V(t,x,y) \le P_2(x) + \frac{a_1}{1+c_2}L(y) = \\ = V_2(x,y) \text{ in } D_2,$$

where $V_i(0,0) = 0$ and $V_i(x,y) > 0$ for every $(x,y) \in R_2 \setminus \{(0,0)\}$ and $i = 1,2$. This means that the function $V(t,x,y)$ is positive definite and by Definition 2 it is uniformly small. Differentiating the function $V(t) = V(t,x(t),y(t))$ with respect to t, using (1.3), (1.12), (1.13), (1.17)–(1.19), (1.21), (1.22) and system (1') we obtain

$$V'(t) \le -\frac{b_i}{1+c_1}\ \frac{y}{l(y)}f(x,y) = -W_3^i(x,y) \le 0 \text{ for } \ i = 1,2 \text{ in } D_2.$$

This means that the function $V'(t,x,y)$ is negative definite, therefore by Proposition 3 the proof is finished.

If we use the function $V(t,x,y)$ from Corollary 1.2., then the next Theorem and Corollary can be proved analogously as Theorem 1.1. and Corollary 1.1., respectively.

Theorem 1.3. *Let* (1.2)–(1.6), (1.8), (1.13), (1.14), (1.16) *and* (1.17) *hold. Moreover, suppose that the following conditions are satisfied:*

(1.23) *There exists a positive constant a_3 such that $a(t) \to a_3$ for $t \to \infty$;*

(1.24) *There exists a positive constant c_3 such that $c(t) \to c_3$ for $t \to \infty$;*

Then all solutions $(x(t), y(t))$ of (1') *are defined and bounded on I.*

Corollary 1.3. *Let the hypotheses of Theorem* 1.3 *and* (1.15) *be fulfilled. Then the trivial solution of* (1') *is stable in the sense of Liapunov.*

The following example is illustrative.

The equation

$$x'' - 2e^{-3t}x^2x' + 2(1+e^{-t})x(x'^2+1) + x'\sqrt{t^2+e^{2t}} = \\ = x'(2x^2 + \sqrt{t^2+x^2} + 2e^t + 3)$$

has a solution $x(t) = e^t$. It is easy to verify that all conditions of Theorem 1.3., except (1.14), are satisfied and $x(t) = e^t$ and $x'(t) = e^t$ are unbounded on I. This means that the condition (1.14) in Theorem 1.3 is substantial. For $g(t,x,y) = 0$ in D_2 we have a further boundedness theorem.

Theorem 1.4. *Let* (1.3)–(1.5), (1.7), (1.8), (1.11), (1.12) *and* (1.16)–(1.21) *hold. Moreover, suppose that the following condition is satisfied:*

(1.25) *$c'(t) \le 0$ for every $t \in I$.*

Then the solutions of (1') *are uniformly bounded for* $t \geq 0$.

PROOF. For any solution $(x(t), y(t))$ of (1') and a positive K_2 we define
$$V(t,x,y) = a(t)L(y) + (1 + c(t))H(t,x) + K_2 \text{ in } D_2.$$
By (1.3), (1.5), (1.8), (1.11), (1.12) and (1.18)–(1.20) we get
$$0 < W_1(x,y) = a_2L(y) + (1+c_2)P_1(x) + K_2 \leq V(t,x,y) \leq$$
$$\leq a_1L(y) + (1+c_1)P_2(x) + K_2 = W_2(x,y)$$
in D_2. From (1.4) and (1.8) it follows $W_1(x,y) \to \infty$ for $|x| \to \infty$ and $|y| \to \infty$.

Differentiating the function $V(t) = V(t,x(t),y(t))$ with respect to t, using (1.3), (1.5)–(1.7), (1.17), (1.21) and (1.25) we obtain
$$V'(t) = a'(t)L(y) + \frac{y}{l(y)}(e(t,x,y) - b(t)f(x,y)) + c'(t)H(t,x) +$$
$$+ \frac{\partial H(t,x)}{\partial t} \leq 0 \text{ in } D_2.$$
This means that all conditions of Proposition 1 are fulfilled, therefore the solutions of (1') are uniformly bounded for $t \geq 0$. This completes the proof.

Corollary 1.4. *Let the hypotheses of Theorem* 1.4. *hold with the exception of* (1.7). *Moreover, suppose that* (1.22) *holds. Then the trivial solution of* (1') *is uniformly asymptotically stable.*

PROOF. From (1.5), (1.21) and (1.22) it follows that $h(t,0) = 0$, $e(t,x,0) = 0$ and $f(x,0) = 0$ for every $t \in I$ and $x \in R_1$. This means that (1') has the trivial solution. For any solution $(x(t), y(t))$ of (1') we define
$$V(t,x,y) = a(t)L(y) + (1 + c(t))H(t,x) \text{ in } D_2. \tag{5}$$
By (1.3), (1.5), (1.8), (1.11), (1.12) and (1.18)–(1.20) we get
$$V_1(x,y) = a_2L(y) + (1+c_2)P_1(x) \leq V(t,x,y) \leq$$
$$\leq a_1L(y) + (1+c_1)P_2(x) = V_2(x,y) \text{ in } D_2,$$
where $V_i(0,0) = 0$ and $V_i(x,y) > 0$ for every $(x,y) \in R_2 \setminus \{(0,0)\}$ and $i = 1,2$. This means that the function $V(t,x,y)$ is positive definite and by Definition 2 it is uniformly small. Differentiating the function $V(t) = V(t,x(t),y(t))$ with respect to t, using (1.3), (1.5), (1.16), (1.17), (1.21), (1.22), (1.25) and (1') we obtain
$$V'(t) \leq -b_i\frac{y}{l(y)}f(x,y) = -W_3^i(x,y) \leq 0 \quad \text{for } i = 1,2 \text{ in } D_2.$$

This means that the function $V'(t,x,y)$ is negative definite, therefore by Proposition 3 the proof is finished.

Using the function (5) we can prove the following boundedness Theorem and its Corollary analogously as Theorem 1.1. and Corollary 1.1.

Theorem 1.5. *Let* (1.3)–(1.6), (1.8), (1.14), (1.16)–(1.18), (1.24) *and* (1.25) *hold.*

Then all solutions $(x(t), y(t))$ *of* (1') *are defined and bounded on* I.

Corollary 1.5. *Let the hypotheses of Theorems* 1.5. *and* (1.15) *hold. Then the trivial solution of* (1') *is stable in the sense of Liapunov.*

Remark 1.1. If we put in (1) $a(t) = 1$, $b(t) = 0$, $g(t,x,x') = 0$, $1 + c(t) = a(t)$, $h(t,x) = f(x)$, $l(x') = g(x')$ and $e(t,x,x') = 0$, then we get Theorem 1 and Theorem 2 and Theorem 3 of [4] from Theorem 1.5 and Theorem 1.3 and Corollary 1.5, respectively.

We are now ready to prove a boundedness and stability theorem for solutions of the equation (1) in the case that $a(t) = 1$ and $c(t) = 0$ in I. Note that in this case the equation (1) is equivalent to the system

$$\begin{aligned} x' &= y, \\ y' &= e(t,x,y) - b(t)f(x,y) - g(t,x,y) - h(t,x)l(y). \end{aligned} \tag{6}$$

Theorem 1.6. *Let* (1.3)–(1.8), (1.17) *and* (1.20) *hold. Moreover, suppose that the following conditions are satisfied:*

(1.26) *There exists a nonnegative function* $r \in C(I)$ *such that* $|e(t,x,y)| \le \frac{1}{2}|y|r(t)$ *and* $\int\limits_0^\infty r(t)\,dt = M < \infty$;

(1.27) $l \in C^1(R_1)$ and $l'(y)\operatorname{sgn} y \ge 0$ *for every* $y \in R_1$.

Then the solutions of (6) *are uniformly bounded for* $t \ge 0$.

PROOF. For any solution $(x(t), y(t))$ of (6) and a positive K_3 we define

$$V(t,x,y) = e^{-E(t)}(H(t,x) + L(y) + K_3) \text{ in } D_2,$$

where $E(t) = \int\limits_0^t r(s)\,ds$. By (1.3)–(1.5), (1.8), (1.20) and (1.26) we get

$$\begin{aligned} 0 < e^{-M}(P_1(x) + L(y) + K_3) = W_1(x,y) \le V(t,x,y) \le \\ \le P_2(x) + L(y) + K_3 = W_2(x,y) \text{ in } D_2 \end{aligned}$$

and $W_1(x,y) \to \infty$ for $|x| \to \infty$ and $|y| \to \infty$. Differentiating $V(t) = V(t, x(t), y(t))$ with respect to t, using (1.6), (1.17) and (1.26) we obtain

$$V'(t) \le -r(t)V(t) + e^{-E(t)}\left(\frac{y^2}{2l(y)}r(t) - b(t)\frac{y}{l(y)}f(x,y)\right).$$

Since $L(y) = \frac{1}{2}\ \frac{y^2}{l(y)} + \frac{1}{2}\int\limits_0^y \frac{s^2 l'(s)}{l^2(s)}\,ds$ for every $y \in R_1$, by (1.5), (1.7), (1.27) and the last inequality we get

$$V'(t) \le -b(t)e^{-E(t)}\ \frac{y}{l(y)} f(x,y) \le 0 \text{ in } I. \tag{7}$$

We have proved that all conditions of Proposition 1 are fulfilled, therefore the solutions of (6) are uniformly bounded for $t \ge 0$. This completes the proof.

Corollary 1.6. *Let the conditions of Theorem* 1.6. *be fulfilled with the exception of* (1.7), *which is replaced by* (1.22). *Then the trivial solution of* (6) *is uniformly asymptotically stable.*

PROOF. From (1.5), (1.6), (1.22) and (1.26) it follows that $h(t,0) = 0$, $g(t,x,0) = 0$, $e(t,x,0) = 0$ and $f(t,0) = 0$ for every $t \in I$ and $x \in R_1$. This means that (6) has the trivial solution. For an arbitrary solution $(x(t), y(t))$ of (6) we define

$$V(t,x,y) = e^{-E(t)}(H(t,x) + L(y)) \text{ in } D_2, \text{ where } E(t) = \int\limits_0^t r(s)\,ds.$$

By (1.3), (1.5), (1.8), (1.20) and (1.26) we get

$$V_1(x,y) = e^{-M}(P_1(x) + L(y)) \le V(t,x,y) \le P_2(x) + L(y) = V_2(x,y)$$

in D_2, where $V_i(0,0) = 0$ and $V_i(x,y) > 0$ for every $(x,y) \in R_2 \setminus \{(0,0)\}$ and $i = 1,2$. Form (7) by (1.22) we obtain

$$V'(t) \le -b_i \frac{y}{l(y)} f(x,y) = -W_3^i(x,y) \le 0,\ i = 1,2 \text{ in } I.$$

With respect to Proposition 2 the proof is complete.

2. The necessary conditions

Let the assumptions of the previous section hold in $D_1^0 = I_0 \times R_1$, resp. $D_2^0 = I_0 \times R_2$, where $I_0 = \langle t_0, \infty)$ for $t_0 \in R_1$.
We have

Theorem 2.1. *Suppose that the assumptions* (1.3), (1.5), (1.6), (1.12) *and* (1.18) *hold. Moreover, suppose that the following conditions are satisfied:*

(2.1) *$b(t) \geq 0$ for every $t \in I_0$ and $yf(x,y) \leq 0$ for every $x, y \in R_1$ or $b(t) \leq 0$ for every $t \in I_0$ and $yf(x,y) \geq 0$ for every $x, y \in R_1$;*

(2.2) *$1 + c(t) \geq 0$ for every $t \in I_0$;*

(2.3) *There exist nonnegative functions $g_1, e_1 \in C(I_0)$ such that $|e(t,x,y)| \leq e_1(t)$, $xe(t,x,y) \geq 0$, $|g(t,x,y)| \leq g_1(t)$ in D_2^0 where $\int\limits_{t_0}^{\infty} e_1(t)\,dt = E_1 < +\infty$ and $\int\limits_{t_0}^{\infty} g_1(t)\,dt = G_1 < +\infty$;*

(2.4) *There exists a positive constant a_4 such that $|f(x,y)| \leq a_4$ for every $x, y \in R_1$;*

(2.5) *$\int\limits_{t_0}^{\infty} |b(t)|\,dt = B_1 < +\infty$;*

(2.6) *$h(t,x)$ is a nonincreasing function in the variable t for every $x > 0$ and a nondecreasing function in the variable t for every $x < 0$.*

If all solutions of (1') are bounded in I_0, then

$$\int\limits_0^{\pm\infty} |h(t_0,x)|dx = \pm\infty.$$

PROOF. Suppose that all solutions $(x(t), y(t))$ of (1') are bounded in I_0 and

$$\int\limits_0^{+\infty} |h(t_0,x)|dx = K_4 < +\infty. \tag{8}$$

We are going to show that there exists a solution of (1'), which is unbounded in I_0. For $l \in C(R_1)$ there exists a positive constant

$$M_1 = \max\{l(y);\ y \in \langle 0,\ 2 + \frac{1}{a_2}(2G_1 + E_1 + a_4B_1)\rangle\}.$$

By (8) there exists $x_0 > 0$ so large that

$$\frac{1+c_1}{a_2} M_1 \int\limits_{x_0}^{+\infty} h(t_0,s)\,ds < 1. \tag{9}$$

We will prove that the solution $(x(t), y(t))$ of (1') with

$$x(t_0) = x_0, \qquad y(t_0) = 2 + \frac{G_1}{a_2} \tag{10}$$

is unbounded for $t \to \infty$.

For $t \geq t_0$ there is $y(t) > 1$. Suppose that this is not the case. Define

$$t_1 = \inf\{t;\ t \geq t_0,\ y(t) \leq 1\}.$$

Now, from the continuity of $y(t)$ and from (10) we obtain

$$t_1 > t_0 \text{ and } y(t) \geq 1 \text{ for } t \in \langle t_0, t_1 \rangle \text{ and } y(t_1) = 1.$$

Hence for $t \in \langle t_0, t_1 \rangle$ the solution $(x(t), y(t))$ lies in the first quadrant. By (1'), (1.3), (1.5), (1.6), (1.12), (1.18), (2.1)–(2.3) and (2.6) we get

$$y'(t) \geq -\frac{1}{a_2} g_1(t) - \frac{1+c_1}{a_2} h(t_0, x) l(y). \tag{11}$$

Integrating the second equation of (1') from t_0 to t ($t \geq t_0$), by (1.3), (1.5), (1.6), (1.12), (1.18) and (2.1)–(2.3) we have

$$y(t) \leq 2 + \frac{1}{a_2}(2G_1 + E_1 + a_1 B_1). \tag{12}$$

Now, integrating (11) from t_0 to t_1, by the Mean Value Theorem and (12) we get

$$y(t_1) \geq 2 - \frac{1+c_1}{a_2} l(y(t_1)) \int\limits_{x_0}^{x(t)} h(t_0, s)\, ds \geq 2 - \frac{1+c_1}{a_2} M_1 \int\limits_{x_0}^{x(t_1)} h(t_0, s)\, ds >$$

$$> 2 - \frac{1+c_1}{a_2} M_1 \int\limits_{x_0}^{\infty} h(t_0, s)\, ds.$$

Therefore by (9) we obtain $y(t_1) > 1$. We have a contradiction. This means that $x'(t) = y(t) > 1$ and $x(t) \geq x_0 + (t - t_0)$ for every $t \geq t_0$. Therefore $x(t) \to +\infty$ for $t \to +\infty$.

A similar argument may be given in the third quadrant in case

$$\int\limits_0^{-\infty} |h(t_0, x)|\, dx = -K_4 > -\infty$$

and that completes the proof.

Remark 2.1. Theorem 2.1 and Theorem 1 in [1] are similar. The authors deal with the system

$$x' + \frac{y}{r(t)}, \qquad y' = -a(t) f(x) g\left(\frac{y}{r(t)}\right),$$

in Theorem 1.

Theorem 2.2. *Suppose that the assumptions* (1.1), (1.3), (1.5), (1.11) *and* (2.1) *hold. Moreover, suppose that the following conditions are satisfied:*

(2.7) *There exists a positive constant c_4 such that $1 + c(t) \geq c_4$ for every $t \in I_0$;*

(2.8) *$yg(t,x,y) \leq 0$ in D_2^0;*

(2.9) *$ye(t,x,y) \geq 0$ in D_2^0;*

(2.10) *$h(t,x)$ is a nonincreasing function in the variable t for every $x < 0$ and a nondecreasing function in the variable t for every $x > 0$;*

(2.11) $\int\limits_0^{\pm\infty} h(t_0,x)\,dx = +\infty$.

If all solutions of (1') *are defined in I_0, then*

$$\int\limits_0^{\pm\infty} \frac{s}{l(s)}\,ds = +\infty\,.$$

Proof. Suppose that all solutions $(x(t), y(t))$ of (1') are defined in I_0 and

$$\int\limits_0^{+\infty} \frac{s}{l(s)}\,ds = K_5 < +\infty. \tag{13}$$

Let $(x(t), y(t))$ be a solution of (1') with

$$x(t_0) = x_0 < 0 \quad \text{and} \quad y(t_0) = 1,$$

where for x_0 we have

$$\frac{c_4}{a_1}\int\limits_0^{x_0} h(t_0,s)\,ds > 2K_5. \tag{14}$$

Since $x_0 < 0$ and $y_0 = y(t_0) > 0$, a part of the solution $(x(t), y(t))$ remains in the second quadrant. Integrating the second equation of (1') from t_0 to t $(t \geq t_0)$, for this part of the solution $(x(t), y(t))$, by (1.1), (1.3), (1.5), (1.11), (2.1) and (2.7)–(2.9), we have $y(t) \geq 1$. This means by (1') that $x(t) \geq x_0 + (t - t_0)$. Therefore for $x_0 < 0$ there exists $t \in I_0$ such that $x(t) = 0$. Further for $t \geq t_0$ we define

$$V(t) = \int\limits_{y_0}^{y(t)} \frac{s}{l(s)}\,ds.$$

Differentiating the last function, by (1.1), (1.3), (1.5), (1.11), (2.1), (2.7)–(2.9) and (2.11), for the part of the solution $(x(t), y(t))$ remaining in the second quadrant we get

$$V'(t) \geq -\frac{c_4}{a_1} y(t)\, h(t_0, x(t)). \tag{15}$$

Integrating (15) from t_0 to t $(t > t_0)$, we have

$$V(t) \geq -\frac{c_4}{a_1} \int_{x_0}^{x(t)} h(t_0, s)\, ds. \tag{16}$$

We prove that $x(t) \neq 0$ for every $t > t_0$. If $x(t) = 0$ for some $t > t_0$, then by (16) for $x(t) = 0$ we get

$$V(t) \geq -\frac{c_4}{a_1} \int_{x_0}^{0} h(t_0, s)\, ds = \frac{c_4}{a_1} \int_{0}^{x_0} h(t_0, s)\, ds.$$

Therefore by (14) we obtain $V(t) > 2K_5$. We have a contradiction with (13), because

$$V(t) = \int_{1}^{y(t)} \frac{s}{l(s)}\, ds < \int_{0}^{+\infty} \frac{s}{l(s)}\, ds = K_5 \quad \text{for } t \geq t_0.$$

Therefore for all solutions $(x(t), y(t))$ of (1'), which are defined in I_0

$$\int_{0}^{+\infty} \frac{s}{l(s)}\, ds = +\infty.$$

A similar argument may be given in the fourth quadrant in case

$$\int_{0}^{-\infty} \frac{s}{l(s)}\, ds < +\infty$$

and the proof is complete.

Remark 2.2. Theorem 2.2 and Theorem 3 in [1] are similar. The authors deal with the system

$$x' = y, \qquad y' = -\frac{r'(t)}{r(t)} y - \frac{a(t)}{r(t)} f(x) g(y)$$

in Theorem 3.

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