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About some positive solutions of the diophantine equation $\sum_{1\le i<j\le n} a_i a_j = m$

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Let $\mathbf{N}$ denote the set of positive integers and let $n, m \in \mathbf{N}$ be fixed. We search for $a_i \in \mathbf{N}$ satisfying the equation

$$\sum_{1\le i<j\le n} a_i a_j = m \tag{1}$$

The case $n = 2$ is trivial. For any m let be $a_1|m$ and $a_2 = m/a_1$. For $n > 3$ we prove that if $m \ge 136n^2$ then (1) has a solution. In the case $n = 3$ the problem is still open. (We have controlled, that (1) has solution for all $m \le 10^7$ except the numbers 1,4,18,22,30,42,58,70,78,102,130,190,210,330 and 462.) Let us use the following notations:

$$A_u^v = \sum_{u\le i\le v} a_i \qquad A_u^{*v} = \sum_{u\le i<j\le v} a_i a_j\,.$$

We prove the following

Theorem. *For $n > 3$ and any $m \ge 136n^2$ the equation $A_1^{*n} = m$ has at least one solution $(a_1, \dots, a_n) \in \mathbf{N}^n$.*

PROOF OF THE THEOREM. We need some Lemmas, which can be easily controlled by computer.

Lemma 1. *For any fixed $q \in \mathbf{N}$ the equations $A_1^3 = 19$ and $a_1^2 - a_2 a_3 \equiv q \pmod{19}$ have a common solution $(a_1, a_2, a_3) \in \mathbf{N}^3$.*

Lemma 2. *For any fixed $q \in \mathbf{N}$, the equations $A_1^3 \doteq 23$ and $A_1^{*3} \equiv q \pmod{26}$ have a common solution $(a_1, a_2, a_3) \in \mathbf{N}^3$.*

Lemma 3. *Let $t \in \{40, 41, \dots, 47\}$ be fixed. Either for all even $q \in \mathbf{N}$ or for all odd $q \in \mathbf{N}$ (depending on t) $A_1^4 = t$ and $A_1^{*4} \equiv q \pmod{52}$ have a common solution $(a_1, a_2, a_3, a_4) \in \mathbf{N}^4$.*

(For example the triplets (1,7,11), (2,5,12), (1,4,14), (3,7,9), (4,5,10), (1,6,12), (3,6,10), (1,2,16), (2,8,9), (2,4,13), (2,7,10), (3,5,11), (1,5,13), (1,3,15), (2,6,11), (5,6,8), (1,8,10), (4,7,8) and (3,4,12) give all the different remainders mod 19 in Lemma 1.)

The case $n = 4$:
Let us write (1) in the form $(m + a_1^2 - a_2a_3)/A_1^3 = a_1 + a_4$. Lemma (1) gives our result for m large enough. (For $n \geq 119$ the a_i's can even be chosen pairwise different.)

The case $n = 5$:
We have $(m + a_1^2 - A_2^{*4})/(a_1 + A_2^4) = a_1 + a_5$. The choice $a_1 = 3$ and $A_2^4 = 23$ gives by Lemma 2 a solution.

The case $n = 6$:
Let be $a_5 = 1$. So (1) ca be written in the form $(m + a_1^2 - A_2^4 - A_2^{*4})/(a_1 + A_2^4 + 1) = a_1 + a_6$. The choice $a_1 = 2$ and $A_2^4 = 23$ gives by Lemma 2 a solution.

The case $n = 7$:
The choice $a_1 = a_2 = a_3 = 1$ and $A_4^6 = 23$ leads by Lemma 2 to a solution, using that (1) implies

$$(m + a_1^2 - (a_2 + a_3)A_4^6 - a_2a_3 - A_4^{*6})/(a_1 + a_2 + a_3 + A_4^6) = a_1 + a_7\,.$$

The cases $8 \leq n \leq 15$:
Let us choose $a_1 = 1, (a_2, a_3) = (1,3)$ or $(2,2)$, $a_i = 1$ for $4 \leq i \leq n-5$ and $A_1^{n-1} = 52$. So $A_{n-4}^{n-1} \in \{40, \ldots, 47\}$, i.e. by Lemma 3 A_{n-4}^{*n-1} can be either in any even or in any odd residue class mod 52. The above choice of the a_i's in (1) gives

$$(2)\quad m + 1 - 4A_4^{n-1} - a_2a_3 - \binom{n-8}{2} - (n-8)A_{n-4}^{n-1} - A_{n-4}^{*n-1} = \\ = (a_1 + a_n)A_1^{n-1}\,.$$

A suitable choice of the pair (a_2, a_3) guarantees that the left side of (2) without A_{n-4}^{*n-1} is even or odd. So we can achieve that the left side of (2) is divisible by 52. For all $n \leq 15$ it can be easily verified, that the choice $m \geq 136n^2$ guarantees $a_n \geq 1$.

The case $n \geq 16$:
Let us choose $a_1 = 22 \cdot 2^{2s} + 13 - n - 23 \cdot 2^s$, $a_{2i} = 2^{2s-1} - c_i$, $a_{2i+1} = 2^{2s-1} + c_i$ with a suitable choice (see later) of $0 \leq c_i < 2^s$ if $1 \leq i \leq 4$, $a_j = 1$ if $10 \leq j \leq n-4$ and $a_t = b_t \cdot 2^s$ with some $b_t > 0$ if $n-3 \leq t \leq n-1$, such that $b_{n-1} + b_{n-2} + b_{n-3} = 23$ is satisfied. So all the a_i's are positive if $2 \leq i \leq n-1$. For a fixed $n \in \mathbf{N}$ all the choices s for which $22 \cdot 2^{2s} - 23 \cdot 2^s + 12 \geq n$ guarantee $a_1 \geq 1$ too. ($s = 1$ is suitable for all $54 \geq n \geq 13$.)

The above choice of the a_i's implies $A_1^{n-1} = 26\cdot 2^{2s}$ and $A_{n-3}^{n-1} = 23\cdot 2^s$. We can write (1) in the form

$$\begin{aligned}(3)\qquad L =& m + a_1^2 - \sum_{i=1}^{4}\left[(a_{2i} + a_{2i+1}) - A_{2i+2}^{n-1} + a_{2i}a_{2i+1}\right] - \\ &- \binom{n-13}{2} - (n-13)A_{n-3}^{n-1} - A_{n-3}^{*n-1} = \\ =&(a_1 + a_n)A_1^{n-1}\,.\end{aligned}$$

To get $a_n \ge 1$ it is sufficient to prove

$$(4)\qquad L \ge (a_1 + 1)A_1^{n-1}$$

and

$$(5)\qquad A_1^{n-1} \mid L\,.$$

If we change the values of the numbers c_i, the left side of (3) remains unchanged excepted the sum

$$\sum_{i=1}^{4} a_{2i}a_{2i+1} = 2^{4s} - \sum_{i=1}^{4} c_i^2\,.$$

By the theorem of Lagrange any positive integer can be written as the sum of at most 4 squares. It is enough to take all summands $0 \le c_i < 2^s$ to guarantee

$$2^{2s} \mid L - A_{n-3}^{*n-1}\,.$$

Lemma 2 implies that for all $q \in \mathbf{N}$ there exist $a_{n-3}, a_{n-2}, a_{n-1}$, such that $A_{n-3}^{n-1} = 23\cdot 2^s$ and $A_{n-3}^{*n-1} \equiv q\cdot 2^{2s} \pmod{26\cdot 2^{2s}}$ are valid. A suitable choice of q and $a_{n-3}, a_{n-2}, a_{n-1}$ implies

$$2^{2s}\cdot 26 \mid L$$

too. So we have (5).

We can choose $22\cdot 2^{2s} - 23\cdot 2^s + 12 \ge n > 22\cdot 2^{2s-2} - 23\cdot 2^{s-1} + 12$ $(s = 1, 2, \dots)$ to get all $n \ge 16$. By (3) and (4), $a_n \ge 1$ can be guaranteed, if we have

$$m \ge 2^{2s}, 26\cdot 2^{2s}, 4 + 2^{4s} + (22\cdot 2^{2s})^2 + 22\cdot 2^{2s}\cdot 23\cdot 2^s + (23\cdot 2^s)^2 + 26\cdot 2^{2s}\cdot 22\cdot 2^{2s}\,.$$

This solves (1) for all $m \ge 1547\cdot 2^{4s}$ too. This condition is satisfied for all $m \ge 97n^2$ if $s \ge 3$ (i.e. for $n \ge 1236$). If $s = 1$ or $s = 2$, the choices

$m \geq 97n^2$ $(54 \geq n \geq 16)$ and $m \geq 136n^2$ $(1235 \geq n > 54)$, respectively, imply $a_n \geq 1$.

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