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The companions of inner mapping groups

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Abstract. The objective of this paper is to investigate the companions of inner mapping groups of some special classes of Moufang loops.
It is shown that
(1) the set of companions of all the inner mappings of a pE loop G is G for $p \neq 3$ and is N, the nucleus, for $p = 3$.
(2) the set of companions of all the inner mappings of an F loop G which is generated by three elements is NG^3G'.

Introduction

For what loop G is it true that every loop isotopic to G is isomorphic to G? This question is of considerable geometric significance, particularly in relation to 3–nets. R. H. Bruck proved in [1, p.64, Theorem 2.3] that a necessary and sufficient condition that every loop isotopic to a Moufang loop G be isomorphic to G is that every element of G be a companion of at least one pseudo-automorphism of G. Since every inner mapping of G is a pseudo-automorphism of G, thus this brings us to the study of the companions of inner mappings of some special classes of Moufang loops.

Definitions

An F loop is a Moufang loop such that if H is a subloop generated by any three elements $x,\ y,\ z$, then the associator $(x,y,z) \in Z(H)$, the centre of H.

A pE loop G is a Moufang loop such that G/N is commutative of exponent p, where N is the nucleus of G and p is a prime.

$I(G)$, the inner mapping group of the loop G is defined as
$\langle R(x,y),\ L(x,y),\ T(x) \,|\, x,y \in G\rangle$ where $R(x,y) = R(x)R(y)R(xy)^{-1}$,
$L(x,y) = L(x)L(y)L(yx)^{-1},\ T(x) = R(x)L(x)^{-1}$.

Define $T(G) = \langle T(x) \,|\, x \in G \rangle$.

A permutation S of a loop G is called a pseudo-automorphism of G provided there exists at least one element c of G, called a companion of S, such that

$$(xS) \cdot (yS.\, c) = (xy)S \cdot c$$

for all x, y in G. If c is a companion of S, then cN is obviously the set of all companions of S. It is known that every element of $I(G)$ is a pseudo-automorphism. Define $C[I(G)]$ and $C[T(G)]$ as the set of companions of $I(G)$ and $T(G)$ respectively.

G_a, the associator subloop of G, is generated by all the associators (x, y, z) where $xy \cdot z = (x \cdot yz)(x, y, z)$.

G_c, the commutator subloop of G, is generated by all the commutators (x, y) where $xy = yx \cdot (x, y)$.

G', the associator-commutator subloop of G, is generated by G_a and G_c.

Facts

Let G be a Moufang loop.

F1. If $G = \langle x, y, z \rangle$ is an F loop, then $G_a = \langle (x, y, z) \rangle \subset Z$.

F2. $S \in I(G) \Rightarrow S$ is a pseudo-automorphism of G [1, p.117, Lemma 3.2].

F3. A companion of $T(x)$ is x^{-3} and a companion of $R(x, y) = L(x^{-1}, y^{-1})$ is (x, y). [1, p.113, Lemma 2.2].

F4. If θ, ψ are pseudo-automorphisms of G with companions a, b respectively, then $\theta\psi$ has companion $(a\psi) \cdot b$ and θ^{-1} has companion $(a\theta^{-1})^{-1}$. [3, p.62, 2(iii)].

F5. If $\theta \in I(G)$, then $\theta = T(g)R(x_1, y_1) \ldots R(x_n, y_n)$ where $g, x_i, y_i \in G$. [2, p.322, Theorem 10A].

F6. $C[T(G)] = NG^3$. [3, p.64, Theoerem 2.2].

F7. If G is an F loop, then $gR(x, y) = g(g, x, y)$, for $g, x, y \in G$. [5, p. 294, Lemma 1].

F8. A pE loop is an F loop. [1, p.125, Lemma 5.5 (ii)].

Bruck's Lemma. *Let G be a Moufang loop. Then G satisfies all or none of the following identifies:*

(i)	$((x, y, z), x) = 1$	*(ii)*	$(x, y, (y, z)) = 1$
(iii)	$(x, y, z)^{-1} = (x^{-1}, y, z)$	*(iv)*	$(x, y, z)^{-1} = (x^{-1}, y^{-1}, z^{-1})$
(v)	$(x, y, z) = (x, zy, z)$	*(vi)*	$(x, y, z) = (x, xy, z)$
(vii)	$(x, y, z) = (x, z, y^{-1})$		

When these identities hold, then the associator (x, y, z) lies in the centre of the subloop generated by x, y, z; and the following identities hold for all integers n :

(viii) $(x, y, z) = (y, z, x) = (y, x, z)^{-1}$

(ix) $(x^n, y, z) = (x, y, z)^n$

(x) $(xy, z) = (x, z)\,((x, z), y)\,(y, z)\,(x, y, z)^3$

Proof. [1, p.125, Lemma 5.5].

Remark. A Moufang loop G is an F loop if and only if G satsifies Bruck's Lemma. [1, p.125, Lemma 5.5].

Theorem 1. *If G is a pE loop with nucleus N, then $G = NG^3$ for $p \neq 3$ and $C[I(G)] = G$.*

Proof. The fact that G/N is commutative of exponent p implies $G_c \subset N$ and $x^p \in N$ for all $x \in G$. As $(p, 3) = 1$, so $p = 3m \pm 1$. Thus, $x^p = x^{3m\pm1} \in N$. Then, $x^{\pm1} = x^p x^{-3m} \in NG^3$. Thus, $G = NG^3$.

As $T(G) \subset I(G)$, therefore $C[T(G)] \subset C[I(G)]$. By F6, $C[T(G)] = NG^3 = G$, so $G \subset C[I(G)]$. Obviously, $C[I(G)] \subset G$, thus $G = C[I(G)]$.

Remark. Since for $p \neq 3$, every element of the pE loop G is a companion of some pseudo-automorphism of G, every isotope of G is isomorphic to G [1, p.115, Theorem 2.3].

Theorem 2. *If G is a $3E$ loop with nucleus N, then $N = NG^3$ and $C[I(G)] = N$.*

Proof. G is a $3E$ loop implies $x^3 \in N$ and thus $G^3 \subset N$. Then $NG^3 = N$. Let $\theta \in I(G)$, then by *F5*, $\theta = T(g)R(x_1, y_1) \ldots R(x_n, y_n)$ where $g, x_i, y_i \in G$ for $i = 1, 2, \ldots, n$.
A companion of

$$
\begin{aligned}
&T(g)R(x_1, y_1) = \\
&= g^{-3}R(x_1, y_1) \cdot (x_1, y_1) = && \text{by } F4 \\
&= g^{-3}(g^{-3}, x_1, y_1) \cdot (x_1, y_1) = && \text{by } F7
\end{aligned}
$$

$$= n_1 \in N \text{ as } G_c \subset N \text{ and } x^3 \in N$$

Then, a companion of

$$\begin{aligned}
&T(g)R(x_1, y_1)R(x_2, y_2) = \\
&= n_1 R(x_2, y_2) \cdot (x_2, y_2) = && \text{by } F4 \\
&= n_1(n_1, x_2, y_2) \cdot (x_2, y_2) = && \text{by } F7 \\
&= n_2 \in N
\end{aligned}$$

Thus, we can deduce that, $C[I(G)] \subset N$. But, $N = NG^3 = C[T(G)] \subset C[I(G)]$. Therefore, $N = C[I(G)]$.

Remark. Commutative Moufang loops are $3E$ loops. Since there exist nonassociative commutative Moufang loops, so an isotope of a $3E$ loop G is not necessarily isomorphic to G by [1, p.58 (ix)]. This distinguishes $3E$ loops from other pE loops.

Theorem 3. *If $G = \langle x, y, z \rangle$ is an F loop, then $NG^3G' = C[I(G)]$.*

Proof. By *F1*, $G_a \subset Z$. Then clearly $NG^3G_c = NG^3G'$. By *F5*, $\theta \in I(G) \Rightarrow \theta = T(g)R(g_1, h_1) \dots R(g_n, h_n);\ g, h_i, g_i \in G,\ i = 1, 2, \dots, n$. The companions of

$$\begin{aligned}
&T(g)R(g_1, h_1) = \\
&= g^{-3}R(g_1, h_1) \cdot (g_1, h_1)N = && \text{by } F4 \\
&= g^{-3}(g^{-3}, g_1, h_1) \cdot (g_1, h_1)N = && \text{by } F7 \\
&= g^{-3}(g_1, h_1)N && \text{as } G_a \subset Z \subset N
\end{aligned}$$

The companions of

$$\begin{aligned}
&T(g)R(g_1, h_1)R(g_2, h_2) = \\
&= [g^{-3}(g_1, h_1)]R(g_2, h_2) \cdot (g_2, h_2)N = && \text{by } F4 \\
&= [g^{-3}(g_1, h_1)](g^{-3}(g_1, h_1), g_2, h_2)(g_2, h_2)N = \\
&= (g^{-3}(g_1, h_1))(g_2, h_2)N = \\
&= g^{-3}[(g_1, h_1)(g_2, h_2)]N && \text{as } G_a \subset Z \subset N
\end{aligned}$$

Similarly, the companions of

$$\begin{aligned}
T(g)R(g_1, h_1) \dots R(g_n, h_n) &= g^{-3}[(g_1, h_1) \dots (g_n, h_n)]N \\
&\subset G^3G_cN = NG^3G_c = NG^3G'
\end{aligned}$$

$\therefore C[I(G)] \subset NG^3G'$.

Conversely, let $g \in NG^3,\ c \in G_c$. Then g is a companion of some $\theta \in T(G)$ by *F6*. Let $c = c_1c_2 \dots c_n$ be associated in some way, where $c_i = (x_i, y_i),\ x_i, y_i \in G,\ i = 1, \dots, n$. By *F4*, we see that gc is a companion of

$$\theta R(x_1, y_1)R(x_2, y_2) \dots R(x_n, y_n)$$

Therefore $NG^3G_c \subset C[I(G)]$

$$\therefore NG^3G_c = C[I(G)].$$

Remark. We do not know whether this result still holds for an F loop with more than three generators.

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