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On locally monomial functions

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Abstract. In the present paper the equation

$$\Delta_y^n f(x) - n!f(y) = o(y^\alpha) \qquad ((x,y) \to (0,0),\ x \le 0 \le x+ny),$$

for real functions, where n is a natural number and α a non-negative real number, is considered.

1. Introduction

The subject of this paper is related to the study of real polynomial and monomial functions with the aid of the Dinghas interval-derivative and the operator $\tilde{D}$ defined below. In the sequel, in the Introduction we assume that f is a real function.

For real numbers x, y write

$$\Delta_y^1 f(x) = f(x+y) - f(x)$$

and, for $n \in \mathbb{N} = \{1, 2, 3, \dots\}$,

$$\Delta_y^{n+1} f(x) = \Delta_y^1(\Delta_y^n f(x)).$$

For a non-negative integer n we say that f is a polynomial function of degree n if $\Delta_y^{n+1} f(x) = 0$ for all $x, y \in \mathbb{R}$; f is called a monomial function

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of degree $n \in \mathbb{N}$ if $\Delta_y^n f(x) = n! f(y)$, $(x, y \in \mathbb{R})$. A monomial function of degree 1 is considered as an additive function, as well. (For polynomial and monomial functions we refer to [10].)

If, for a positive integer n and for a real number ξ, the limit

$$D^n f(\xi) := \lim_{\substack{(x,y)\to(\xi,0)\\ x\le\xi\le x+ny}} \frac{\Delta_y^n f(x)}{y^n}$$

exists, then $D^n f(\xi)$ is said to be the n^{th} Dinghas interval-derivative of f at ξ (cf. [1]). We consider, furthermore, the operator

$$\tilde{D}^n f(\xi) := \lim_{\substack{(x,y)\to(\xi,0)\\ x\le\xi\le x+ny}} \frac{\Delta_y^n f(x) - n! f(y)}{y^n},$$

as far as it exists.

Polynomial and monomial functions can be characterized by the operators above: A. Simon and P. Volkmann proved in [6] that for a non-negative integer n, a function is a polynomial function of degree n if and only if its $(n+1)^{\text{th}}$ Dinghas derivative is zero at all $\xi \in \mathbb{R}$. It was shown in [2] that for a positive integer n, a function f is a monomial function of degree n if and only if $\tilde{D}^n f(\xi) = 0$ for all $\xi \in \mathbb{R}$. It was also proved in [2] that for $n \in \mathbb{N}$, the property $\tilde{D}^n f(0) = 0$ implies $f(ly) - l^n f(y) = o(y^n)$, $(y \searrow 0)$ for any integer l.

The investigation of the local properties of the operators D and $\tilde{D}$ are motivated by the result mentioned above. The following two problems in this field are due to P. Volkmann: given $n \in \mathbb{N}$, does the property $D^{n+1} f(0) = 0$ imply that there exists a polynomial function $p : \mathbb{R} \to \mathbb{R}$ of degree n such that $f(z) - p(z) = o(z^n)$, $(z \to 0)$; and similarly does $\tilde{D}^n f(0) = 0$ imply that there exists a monomial function $g : \mathbb{R} \to \mathbb{R}$ of degree n such that $f(z) - g(z) = o(z^n)$, $(z \to 0)$? A. Simon and P. Volkmann in [7] gave a positive answer to the first question in the case when $n = 1$. Furthermore, they proved the following more general theorem: for an arbitrary non-negative real number $\alpha \neq 1$ if

$$\lim_{\substack{(x,y)\to(0,0)\\ x\le 0\le x+2y}} \frac{\Delta_y^2 f(x)}{y^\alpha} = 0,$$

then there exists a polynomial function $p : \mathbb{R} \to \mathbb{R}$ of degree 1 such that $f(z) - p(z) = o(|z|^\alpha)$, $(z \to 0)$.

Surprisingly, the answer to the question related to the operator $\tilde{D}^n f(0)$ is negative. A counterexample is given by $F : (-1, 1) \to \mathbb{R}$, $F(x) = x \ln(-\ln|x|)$ for $x \neq 0$, $F(0) = 0$. (See [3] and [7].) In the present paper the relation

$$\lim_{\substack{(x,y)\to(0,0)\\ x\le 0\le x+ny}} \frac{\Delta_y^n f(x) - n!f(y)}{y^\alpha} = 0,$$

or in other words

$$\Delta_y^n f(x) - n!f(y) = o(y^\alpha) \quad ((x,y)\to(0,0),\ x \le 0 \le x+ny) \tag{1}$$

is studied (it is strongly related to some results in [7]), and a function f satisfying (1) is called a locally monomial function of degree n with order α, at 0.

In the second part of the paper we show that if, for $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\alpha > n$, a function f is a locally monomial function of degree n with order α, at 0, then there exists a monomial function $g : \mathbb{R} \to \mathbb{R}$ of degree n such that

$$f(x) - g(x) = o(|x|^\alpha) \quad (x \to 0). \tag{2}$$

For some similar results on monomial functions of degree 1 and 2 we refer to [8] and [9].

In the third part of the paper we prove that if f is a locally monomial function of degree 1 with order α (i.e. a locally additive function with order α), at 0, then even for $0 \le \alpha < 1$ there exists a monomial function $g : \mathbb{R} \to \mathbb{R}$ of degree 1 (i.e. an additive function), such that (2) holds.

The results in the paper lead to the conjecture that for an arbitrary $n \in \mathbb{N}$, $\alpha \ge 0$, $\alpha \neq n$ if the function f satisfies (1) then there exists a monomial function of degree n with property (2), but it may occur that exactly when $\alpha = n$ (i.e. in the case of the operator $\tilde{D}$) there exists no such monomial function.

2. Locally monomial functions of degree n with order $\alpha > n$

Lemma 1. *For $n, \lambda \in \mathbb{N}$, $\lambda \geq 2$ put*

$$A = \begin{pmatrix} \alpha_0^{(0)} & \dots & \alpha_0^{(\lambda n)} \\ \vdots & \ddots & \vdots \\ \alpha_{(\lambda-1)n}^{(0)} & \dots & \alpha_{(\lambda-1)n}^{(\lambda n)} \end{pmatrix},$$

where for $i = 0, \dots, (\lambda - 1)n$ and $k = -i, \dots, \lambda n - i$

$$\alpha_i^{(i+k)} = \begin{cases} (-1)^k \binom{n}{n-k}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Let a_i denote the i^{th} row in A, $(i = 0, \dots, (\lambda - 1)n)$. Furthermore, let $b = (\beta^{(0)} \dots \beta^{(\lambda n)})$, where

$$\beta^{(k)} = \begin{cases} (-1)^{\frac{k}{\lambda}} \binom{n}{n - \frac{k}{\lambda}}, & \text{if } \lambda \mid k \\ 0, & \text{if } \lambda \nmid k \end{cases}$$

for $k = 0, \dots, \lambda n$.

There are positive integers $K_0, \dots, K_{(\lambda-1)n}$ such that

$$K_0 a_0 + \ldots + K_{(\lambda-1)n} a_{(\lambda-1)n} = b, \tag{3}$$

and

$$K_0 + \ldots + K_{(\lambda-1)n} = \lambda^n. \tag{4}$$

Proof. It is trivial that the lemma holds for $n = 1$, $\lambda \geq 2$, $\lambda \in \mathbb{N}$ with $K_0 = \dots = K_{\lambda-1} = 1$.

For $n, \lambda \geq 2$, $n, \lambda \in \mathbb{N}$ the existence of positive integers satisfying (3) was proved in Lemma 2 in [3]. The numbers $K_0, \dots, K_{(\lambda-1)n}$ satisfy

$$(1 + x + \ldots + x^{\lambda-1})^n = K_0 + K_1 x^1 + \ldots + K_{(\lambda-1)n} x^{(\lambda-1)n} \quad (x \in \mathbb{R}),$$

therefore, substituting $x = 1$ we get (4).

Theorem 1. *Let* $\alpha \geq 0$ *be a real,* n *be an arbitrary natural number and* f *be a real function with property* (1). *Then we have*

$$f(lz) - l^n f(z) = o(|z|^\alpha) \quad (z \to 0). \tag{5}$$

for any integer l.

Proof. In the special case $\alpha = n$ Theorem 1 was proved in [2]. The proof, given here, is similar, with some technical simplifications.

Let $\alpha \geq 0$ and $n \in \mathbb{N}$ be given numbers and let $f : \mathbb{R} \to \mathbb{R}$ satisfy (1). We show relation (5) in two steps.

I. At first we prove, by induction on l, that (1) implies

$$f(lz) = l^n f(z) + o(z^\alpha) \quad (z \searrow 0) \tag{6}$$

for any $l \in \mathbb{N}$.

The case $l = 1$ is trivial.

Let $l > 1$ be an arbitrary integer and suppose that

$$f(jy) - j^n f(y) = o(y^\alpha) \quad (y \searrow 0) \tag{7}$$

has already been proved for $j = 1, \ldots, l-1$.

We define the real functions $\varepsilon_0, \ldots, \varepsilon_{(l-1)n}$ and ε as follows:

$$\varepsilon_i(z) := \Delta_z^n f(-iz) - n! f(z) \quad (i = 0, \ldots, (l-1)n; z \in \mathbb{R}) \tag{8}$$

and

$$\varepsilon(z) := \Delta_{lz}^n f(-(l-1)nz) - n! f(lz) \quad (z \in \mathbb{R}). \tag{9}$$

Using the notation of Lemma 1 for $\lambda = l$ and by the well-known formula

$$\Delta_y^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + ky) \quad (x, y \in \mathbb{R})$$

we get that these equations can be written as

$$\varepsilon_i(z) = \sum_{k=0}^{ln} \alpha_i^{(k)} f((n-k)z) - n! f(z) \\ (i = 0, \ldots, (l-1)n;\ z \in \mathbb{R}) \tag{10}$$

and

$$\varepsilon(z)=\sum_{k=0}^{ln}\beta^{(k)}f((n-k)z)-n!f(lz)\quad(z\in\mathbb{R}).$$

By Lemma 1 there exist positive integers $K_1,\dots,K_{(l-1)n}$ for which

$$K_0a_0+\ldots+K_{(l-1)n}a_{(l-1)n}-b=0$$

and

$$K_0+\ldots+K_{(l-1)n}=l^n.$$

Therefore, by the equations in (8) and (9) we obtain

$$n!\Big(f(lz)-l^nf(z)\Big)=K_0\varepsilon_0(z)+\ldots+K_{(l-1)n}\varepsilon_{(l-1)n}(z)-\varepsilon(z)\quad(z\in\mathbb{R}).$$

To prove (6), we show that for $k=0,\dots,(l-1)n$

$$\varepsilon_k(z)=o(z^\alpha)\quad(z\searrow 0)\tag{12}$$

and

$$\varepsilon(z)=o(z^\alpha)\quad(z\searrow 0).\tag{13}$$

If we choose $x=-(l-1)nz$ and $y=lz$ for $z>0$, $z\in\mathbb{R}$, then $x\le 0\le x+ny$, so (1) and (9) imply (13).

If we replace (x,y) by

$$(0,z),(-z,z),\dots,(-nz,z)\quad(z\in\mathbb{R},\ z>0),$$

then $x\le 0\le x+ny$, therefore, from (1) and (8) we have (12) for $k=0,\dots,n$. In the case $l=2$ property (12) is already proved. If $l>2$ for $k=0,\dots,(l-1)n$ we prove it by induction on k. The proof is done for $0\le k\le n$. Let $n<k\le(l-1)n$ be an arbitrary fixed integer and suppose that

$$\varepsilon_r(z)=o(z^\alpha)\quad(z\searrow 0)\tag{14}$$

is true for $r=0,\dots,k-1$. Set

$$\tilde{l}=\left[\frac{k-1}{n}\right]+1,$$

where [ ] denotes the integer part of a real number and we define $\tilde{\varepsilon}:\mathbb{R}\to\mathbb{R}$ as follows:

$$\tilde{\varepsilon}(z) := \Delta^n_{\tilde{l}z} f(-kz) - n! f(\tilde{l}z). \tag{15}$$

Since

$$k-n \le n\left[\frac{k-1}{n}\right]$$

for $x=-kz$ and $y=\tilde{l}z$, we have $x\le 0$ and

$$x+ny = -kz + n\left(\left[\frac{k-1}{n}\right]+1\right)z \ge 0,$$

hence (1) implies

$$\tilde{\varepsilon}(z) = o(z^\alpha) \quad (z \searrow 0). \tag{16}$$

Let $c = (\gamma_0,\dots,\gamma_{n+k})$ be a vector with components $\gamma_0 = \dots = \gamma_{n+k-\tilde{l}n-1} = 0$ and write

$$\gamma_{n+k-\tilde{l}n+j} = \begin{cases} (-1)^{\frac{j}{\tilde{l}}}\binom{n}{n-\frac{j}{\tilde{l}}}, & \text{if } \tilde{l} \mid j \\ 0, & \text{otherwise} \end{cases}$$

for $j=0,\dots,\tilde{l}n$. The simple inequality

$$\left[\frac{k-1}{n}\right]n \le k-1$$

yields

$$\begin{aligned} n+k-\tilde{l}n &= n+k-\left[\frac{k-1}{n}\right]n - n \\ &\ge k-(k-1) = 1, \end{aligned} \tag{17}$$

and then the components $\gamma_{n+k-\tilde{l}n},\gamma_{n+k-\tilde{l}n+1},\dots,\gamma_{n+k}$ of the vector c, defined above, exist.

It is easy to see, like in (10) and (11), that (15) can be written in the following form:

$$\tilde{\varepsilon}(z) = \sum_{j=0}^{n+k} \gamma_j f((n-j)z) - n! f(\tilde{l}z). \tag{18}$$

Let us omit the components $\gamma_0, \dots, \gamma_{n+k-\tilde{l}n-1}$ of the vector c and denote the resulting vector by $\tilde{b} = (\tilde{\beta}^{(0)} \dots \tilde{\beta}^{(\tilde{l}n)})$. It can be seen from the definition of c that $\tilde{b}$ equals $b = (\beta^{(0)} \dots \beta^{(\tilde{l}n)})$ which was given for n and $\lambda = \tilde{l}$ in Lemma 2.2. It is also easy to see, since we have cancelled only zeroes from c, that (18) can be written as follows:

$$\tilde{\varepsilon}(z) = \sum_{j=0}^{\tilde{l}n} \tilde{\beta}^{(j)} f(n-(n+k-\tilde{l}n+j)z) - n! f(\tilde{l}z) \quad (z \in \mathbb{R}). \tag{19}$$

Let us now consider the functions $\varepsilon_{n+k-\tilde{l}n}, \varepsilon_{n+k-\tilde{l}n+1}, \dots, \varepsilon_k$ and the corresponding coefficient vectors $a_{n+k-\tilde{l}n}, a_{n+k-\tilde{l}n+1}, \dots, a_k$ from (10). It follows from the definition of these vectors (see Lemma 1) that for their components $i = n+k-\tilde{l}n,\ n+k-\tilde{l}n+1, \dots, k$

$$\alpha_i^{(0)} = \alpha_i^{(1)} = \dots = \alpha_i^{n+k-\tilde{l}n-2} = \alpha_i^{n+k-\tilde{l}n-1} = 0.$$

If we omit these components from these vectors and denote them, in the order above, by

$$\begin{gathered} \tilde{a}_0 = (\tilde{\alpha}_0^{(0)} \dots \tilde{\alpha}_0^{(\tilde{l}n)}) \\ \vdots \\ \tilde{a}_{(\tilde{l}-1)n} = (\tilde{\alpha}_{(\tilde{l}-1)n}^{(0)} \dots \tilde{\alpha}_{(\tilde{l}-1)n}^{(\tilde{l}n)}), \end{gathered}$$

then we can write the functions $\varepsilon_{n+k-\tilde{l}n}, \varepsilon_{n+k-\tilde{l}n+1}, \dots, \varepsilon_k$ in the form

$$\begin{aligned} \varepsilon_{n+k-\tilde{l}n}(z) &= \sum_{s=0}^{\tilde{l}n} \tilde{\alpha}_0^{(s)} f(n-(n+k-\tilde{l}n+s)z) - n! f(z) \\ &\vdots \\ \varepsilon_k(z) &= \sum_{s=0}^{\tilde{l}n} \tilde{\alpha}_{(\tilde{l}-1)n}^{(s)} f(n-(n+k-\tilde{l}n+s)z) - n! f(z). \end{aligned} \tag{20}$$

One can see that $\tilde{a}_0, \dots, \tilde{a}_{(\tilde{l}-1)n}$ are equal to the vectors

$$\begin{aligned} a_0 &= (\alpha_0^{(0)} \dots \alpha_0^{(\tilde{l}n)}) \\ &\vdots \\ a_{(\tilde{l}-1)n} &= (\alpha_{(\tilde{l}-1)n}^{(0)} \dots \alpha_{(\tilde{l}-1)n}^{(\tilde{l}n)}), \end{aligned}$$

defined for n and $\lambda = \tilde{l}$ in Lemma 1. So by this lemma, there exist positive integers $\tilde{K}_0, \dots, \tilde{K}_{(\tilde{l}-1)n}$ such that $\tilde{K}_0\, \tilde{a}_0 + \dots + \tilde{K}_{(\tilde{l}-1)n}\, \tilde{a}_{(\tilde{l}-1)n} - \tilde{b} = 0$ and $\tilde{K}_0 + \dots + \tilde{K}_{(\tilde{l}-1)n} = \tilde{l}^n$. Thus (19) and (20) imply

$$\begin{aligned} &\tilde{K}_0 \varepsilon_{n+k-\tilde{l}n}(z) + \tilde{K}_1 \varepsilon_{n+k-\tilde{l}n+1}(z) + \dots + \tilde{K}_{(\tilde{l}-1)n} \varepsilon_k(z) \\ &\qquad = \tilde{\varepsilon}(z) + n! f(\tilde{l}z) - n! \tilde{l}^n f(z) \qquad (z \in \mathbb{R}), \end{aligned}$$

that is

$$\begin{aligned} \varepsilon_k(z) = -\frac{1}{\tilde{K}_{(\tilde{l}-1)n}} \Big(&\tilde{K}_0 \varepsilon_{n+k-\tilde{l}n}(z) + \tilde{K}_1 \varepsilon_{n+k-\tilde{l}n+1}(z) + \dots \\ \dots + \tilde{K}_{(\tilde{l}-1)n-1} &\varepsilon_{k-1}(z) + \tilde{\varepsilon}(z) + n! \Big(f(\tilde{l}z) - \tilde{l}^n f(z) \Big) \Big) \qquad (z \in \mathbb{R}). \end{aligned} \tag{21}$$

From

$$\tilde{l} = \left[\frac{k-1}{n}\right] + 1 \le \frac{k-1}{n} + 1 \le \frac{(l-1)n - 1 + n}{n} < l$$

together with the inductive hypothesis (7) we get:

$$f(\tilde{l}z) - \tilde{l}^n f(z) = o(z^\alpha) \quad (z \searrow 0).$$

By (14) and (17)

$$\varepsilon_r(z) = o(z^\alpha) \quad (z \searrow 0)$$

for $r = n + k - \tilde{l}n, \dots, k-1$. Combining (21), (16) and the previous two formulae we get

$$\varepsilon_k(z) = o(z^\alpha) \quad (z \searrow 0).$$

II. Now we prove that under our assumptions $f(0)=0$ and

$$f(-z)-(-1)^n f(z)=o(|z|^\alpha) \quad (z\to 0). \tag{22}$$

We consider the functions

$$\varepsilon_0(z)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(kz)-n!f(z)$$

and

$$\varepsilon_1(z)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f((k-1)z)-n!f(z),$$

defined in (8). By the well-known formulae

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^n-n!=0$$

and

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(k-1)^n-n!=0$$

we can write the functions ε_0 and ε_1 in the form

$$\varepsilon_0(z)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\Big(f(kz)-k^n f(z)\Big) \tag{23}$$

and

$$\varepsilon_1(z)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\Big(f((k-1)z)-(k-1)^n f(z)\Big). \tag{24}$$

In the first part of the proof we have shown that $\varepsilon_0(z)=o(z^\alpha)$, $\varepsilon_1(z)=o(z^\alpha)$ and $f(lz)-l^n f(z)=o(z^\alpha)$, $(z\searrow 0,\ l=1,\dots,n)$. This relation together with (23) implies $f(0)=0$, therefore, applying (24) we get

$$f(-z)-(-1)^n f(z)=o(z^\alpha) \quad (z\searrow 0)$$

which yields (22).

Finally, (6) and (22) prove Theorem 1.

Theorem 2. *Let $\delta > 0$ be a real number and $n \in \mathbb{N}$. If the function $f : [-\delta, \delta] \to \mathbb{R}$ satisfies the property*

$$\Delta_y^n f(x) - n! f(y) = 0 \quad (x \in [-\delta, 0],\ y, x + ny \in [0, \delta]), \tag{25}$$

then for any integer l there exists a real number $\delta_l > 0$ such that

$$f(lz) - l^n f(z) = 0 \quad (z \in [-\delta_l, \delta_l]). \tag{26}$$

Proof. Let $\delta > 0$ and $n \in \mathbb{N}$ be given and let $f : [-\delta, \delta] \to \mathbb{R}$ be a function satisfying (25). We prove that for an arbitrary integer l with $\delta_l = \frac{\delta}{|l|n}$ equation (26) holds.

The proof can be done in a similar way as in the proof of Theorem 1, therefore, we give the outline of the argument, only.

At first, we show by induction on l that for any $l \in \mathbb{N}$

$$f(lz) - l^n f(z) = 0 \quad \Big(z \in \Big[-\frac{\delta}{ln}, \frac{\delta}{ln}\Big]\Big). \tag{27}$$

For $l > 1$ we define the functions

$$\varepsilon_0, \dots, \varepsilon_{(l-1)n} \text{ and } \varepsilon : \Big[-\frac{\delta}{ln}, \frac{\delta}{ln}\Big] \to \mathbb{R}$$

by the same formula as in (8) and (9) and we use a similar method as in the proof of Theorem 1, to show that

$$\varepsilon_k(z) = 0 \quad \Big(z \in \Big[-\frac{\delta}{ln}, \frac{\delta}{ln}\Big]\Big)$$

for $k = 0, \dots, (l-1)n$ and

$$\varepsilon(z) = 0 \quad \Big(z \in \Big[-\frac{\delta}{ln}, \frac{\delta}{ln}\Big]\Big).$$

By Lemma 1

$$n!\Big(f(lz) - l^n f(z)\Big) = K_0\varepsilon_0(z) + \ldots + K_{(l-1)n}\varepsilon_{(l-1)n}(z) - \varepsilon(z) \qquad \Big(z \in \Big[-\frac{\delta}{ln}, \frac{\delta}{ln}\Big]\Big),$$

which proves (27).

To prove

$$f(-z)-(-1)^n f(z)=0 \quad \Big(z\in\Big[-\frac{\delta}{n},\frac{\delta}{n}\Big]\Big) \tag{28}$$

we consider the functions ε_0 and ε_1 on the interval $\Big[-\frac{\delta}{n^2},\frac{\delta}{n^2}\Big]$. Here we have

$$\varepsilon_0(z)=0 \quad \Big(z\in\Big[-\frac{\delta}{n^2},\frac{\delta}{n^2}\Big]\Big)$$

and

$$\varepsilon_1(z)=0 \quad \Big(z\in\Big[-\frac{\delta}{n^2},\frac{\delta}{n^2}\Big]\Big),$$

therefore, we get, by the method used in the second part of the proof of Theorem 1, that

$$f(-z)-(-1)^n f(z)=0 \quad \Big(z\in\Big[-\frac{\delta}{n^2},\frac{\delta}{n^2}\Big]\Big),$$

from which with

$$f(2z)-2^n f(z)=0 \quad \Big(z\in\Big[-\frac{\delta}{2n},\frac{\delta}{2n}\Big]\Big)$$

(28) follows.

Finally, (28) together with (27) implies (26).

Theorem 3. *Let $\delta>0$ be a real number and $n\in\mathbb{N}$. If the function $f:[-\delta,\delta]\to\mathbb{R}$ satisfies property* (25)*, then there exists a real number $\bar{\delta}>0$ such that*

$$\Delta_y^n f(x)-n!f(y)=0 \tag{29}$$

for $x,y,x+ny\in[-\bar{\delta},\bar{\delta}]$.

Proof. Let $\delta>0$ and $n\in\mathbb{N}$ be given numbers and let $f:[-\delta,\delta]\to\mathbb{R}$ be a function with property (25). Let, furthermore, $\bar{\delta}=\frac{\delta}{2n}$ and $\bar{x}$ and $\bar{y}$ be fixed numbers for which $\bar{x},\bar{y},\bar{x}+n\bar{y}\in[-\bar{\delta},\bar{\delta}]$.

It is trivial, that in the case when $\bar{y}=0$ equation (29) holds. For an arbitrary function $\varphi:\mathbb{R}\to\mathbb{R}$ we have the simple formula

$$\Delta_y^n\varphi(x)=(-1)^n\Delta_{-y}^n\varphi(x+ny) \quad (x,y\in\mathbb{R}),$$

so by

$$f(-y)-(-1)^n f(y)=0 \quad (y\in[-\bar{\delta},\bar{\delta}]),$$

which was proved in Theorem 2, we can write

$$\Delta_y^n f(x) - n!f(y) = (-1)^n\big(\Delta_{-y}^n f(x+ny) - n!f(-y)\big)$$
$$(x,\, y,\, x+ny \in [-\bar\delta, \bar\delta]).$$

Therefore, we may suppose that $\bar y \in (0, \bar\delta]$.

In the case when $\bar x \in [-\bar\delta, 0]$ and $\bar x + n\bar y \in [0, \bar\delta]$, (29) comes from (1). If $\bar x, \bar x + n\bar y \in [-\bar\delta, 0]$ and $\bar y \in (0, \bar\delta]$ since

$$\begin{aligned}\Delta_y^n f(x) - n!f(y) &= (-1)^n\Delta_{-y}^n f(-x) - (-1)^n n!f(-y)\\ &= \Delta_y^n f(-x-ny) - n!f(y) \qquad (x, y, x+ny \in [-\bar\delta, \bar\delta])\end{aligned}$$

with $\tilde x = -\bar x - n\bar y$ we get $\tilde x, \bar y, \tilde x + n\bar y \in (0, \bar\delta]$, which means that we may suppose that $\bar x \in (0, \bar\delta]$. Therefore, it is sufficient to prove (29) for $\bar x, \bar y \in (0, \bar\delta]$.

If $\bar x$ and $\bar y$ have these properties, then there exist natural numbers m such that $\bar x - m\bar y \le 0$. Let m_0 be the smallest natural number with this property and we define $x_\mu^* = \bar x - (m_0 - \mu)\bar y$ for $\mu = 0, \dots, m_0$.

We prove by induction on μ that by

$$c_\mu := \Delta_{\bar y}^n f(x_\mu^*) - n!f(\bar y) \tag{30}$$

$c_\mu = 0$ for $\mu = 0, \dots, m_0$, which with $\mu = m_0$ implies

$$\Delta_{\bar y}^n f(\bar x) - n!f(\bar y) = 0,$$

which is our statement.

By (25), obviously, $c_0 = 0$.

Let $\mu \in \{1, \dots, m_0\}$ and suppose that $c_\nu = 0$ is already proved for $\nu = 0, \dots, \mu - 1$. Taking

$$x = x_\mu^* - i\bar y, \quad y = \bar y \quad (i = 1, \dots, n)$$

and

$$x = x_\mu^* - n\bar y, \quad y = 2\bar y,$$

respectively, the inductive hypothesis and (25) lead to

$$\Delta_{\bar y}^n f(x_\mu^* - i\bar y) - n!f(\bar y) = 0 \quad (i = 1, \dots, n)$$

and

$$\Delta_{2\bar y}^n f(x_\mu^* - n\bar y) - n!f(2\bar y) = 0.$$

It is easy to see that with the notation of Lemma 1 (for $\lambda = 2$) we can write these equations as follows

$$(31) \qquad \sum_{k=0}^{2n} \alpha_i^{(k)} f(x_\mu^* + (n-k)\bar{y}) - n! f(\bar{y}) = 0 \quad (i = 1, \dots, n)$$

and

$$(32) \qquad \sum_{k=0}^{2n} \beta^{(k)} f(x_\mu^* + (n-k)\bar{y}) - n! f(2\bar{y}) = 0.$$

Furthermore, (30) has the form

$$(33) \qquad \sum_{k=0}^{2n} \alpha_0^{(k)} f(x_\mu^* + (n-k)\bar{y}) - n! f(\bar{y}) = c_\mu.$$

By Lemma 1 for $a_i = (\alpha_i^{(0)}, \dots, \alpha_i^{(2n)})$, $(i = 0, \dots, n)$ and $b = (\beta^{(0)}, \dots, \beta^{(2n)})$ there exist positive integers $K_0, \dots, K_n$ such that $K_0 a_0 + \ldots + K_n a_n - b = 0$ and $K_0 + \cdots + K_n = 2^n$. Therefore, by the equations in (31), (32) and (33) we get

$$-(K_0 + \cdots + K_n) n! f(\bar{y}) + n! f(2\bar{y}) = K_0 c_\mu,$$

that is

$$-2^n n! f(\bar{y}) + n! f(2\bar{y}) = K_0 c_\mu.$$

By Theorem 2 we have $f(2\bar{y}) - 2^n f(\bar{y}) = 0$, which implies $c_\mu = 0$.

Theorem 4. *Let n be a natural number and $\alpha > n$ be a real number. If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies*

$$(1) \qquad \Delta_y^n f(x) - n! f(y) = o(y^\alpha) \quad ((x, y) \to (0, 0),\ x \le 0 \le x + ny)$$

then there exists a monomial function $g : \mathbb{R} \to \mathbb{R}$ of degree n such that

$$(2) \qquad f(x) - g(x) = o(|x|^\alpha) \quad (x \to 0).$$

Proof. Let $n \in \mathbb{N}$ and $\alpha > n$, $\alpha \in \mathbb{R}$ be given. For a function $f : \mathbb{R} \to \mathbb{R}$ satisfying (1) Theorem 1 implies

$$(34) \qquad f(lz) - l^n f(z) = o(|z|^\alpha) \quad (z \to 0)$$

for any integer l. Let now $l \in \mathbb{N}$, $l > 1$ be fixed. It is easy to see, that (34) is equivalent to the following statement: there exist a real number $\delta > 0$ and a continuous, increasing function $h : [0, \delta] \to \mathbb{R}$ with the property $\lim_{z \searrow 0} h(z) = 0$ such that

$$|f(lz) - l^n f(z)| \le |z|^\alpha h(|z|) \quad (z \in [-\delta, \delta]).$$

Therefore, for an arbitrary $z_0 \in [-\delta, \delta]$ and $k \in \mathbb{N}$ we have

$$\left| f\left(\frac{z_0}{l^{k-1}}\right) - l^n f\left(\frac{z_0}{l^k}\right) \right| \le \frac{|z_0|^\alpha}{l^{k\alpha}} h\left(\frac{|z_0|}{l^k}\right).$$

With

$$\varepsilon_k(z_0) := l^{(k-1)n} f\left(\frac{z_0}{l^{k-1}}\right) - l^{kn} f\left(\frac{z_0}{l^k}\right)$$

we get

$$|\varepsilon_k(z_0)| \le l^{(k-1)n} \frac{|z_0|^\alpha}{l^{k\alpha}} h\left(\frac{|z_0|}{l^k}\right)$$

and the monotony of h yields

$$|\varepsilon_k(z_0)| \le \frac{1}{l^{k(\alpha-n)}} \frac{|z_0|^\alpha}{l^n} h(|z_0|). \tag{35}$$

For an arbitrary $N \in \mathbb{N}$ we obtain

$$\varepsilon_1(z_0) + \cdots + \varepsilon_N(z_0) = f(z_0) - l^{Nn} f\left(\frac{z_0}{l^N}\right). \tag{36}$$

Since $\alpha > n$

$$\sum_{k=1}^{\infty} \frac{1}{l^{k(\alpha-n)}} = \frac{1}{l^{\alpha-n} - 1},$$

therefore,

$$\sum_{k=1}^{\infty} \varepsilon_k(z_0)$$

is convergent, so the limit

$$g(z_0) = \lim_{k\to\infty} l^{kn} f\left(\frac{z_0}{l^k}\right) \tag{37}$$

exists, and (35) and (36) yield

$$|f(z_0) - g(z_0)| \le \frac{1}{l^{\alpha-n} - 1} \frac{|z_0|^\alpha}{l^n} h(|z_0|),$$

which implies (2).

For $x \in [-\delta, 0]$, $x + ny \in [0, \delta]$ by (1) we have

$$\lim_{k\to\infty} \frac{\Delta^n_{\frac{y}{l^k}} f\left(\frac{x}{l^k}\right) - n! f\left(\frac{y}{l^k}\right)}{\left(\frac{y}{l^k}\right)^\alpha} = 0,$$

and (37) gives

$$\Delta^n_y g(x) - n! g(y) = \lim_{k\to\infty} l^{kn} \left(\Delta^n_{\frac{y}{l^k}} f\left(\frac{x}{l^k}\right) - n! f\left(\frac{y}{l^k}\right)\right) = 0,$$

which together with Theorem 3 show that there exists a real number $\bar{\delta} > 0$ such that g is a monomial function of degree n on the interval $[-\bar{\delta}, \bar{\delta}]$. This result and the known extension theorem for monomial functions (cf. [5], for instance) imply our statement.

3. Locally additive functions with order $\alpha \neq 1$

Lemma 2. *Let δ be a positive real number and $f : [-\delta, \delta] \to \mathbb{R}$. If there exists a real number $K \ge 0$ such that*

$$|f(x+y) - f(x) - f(y)| \le K \quad (x \in [-\delta, 0],\ y, x+y \in [0, \delta]), \tag{38}$$

then we have

$$|f(x+y) - f(x) - f(y)| \le 3K \tag{39}$$

for all x, y, $x + y \in [-\delta, \delta]$.

Proof. Let $\bar{x}$ and $\bar{y}$ be fixed real numbers such that $\bar{x}$, $\bar{y}$, $\bar{x} + \bar{y} \in [-\delta, \delta]$. Then we have one of the following relations:

(A) $\bar{x} \in [-\delta, 0],\ \bar{y} \in [0, \delta],\ \bar{x} + \bar{y} \in [0, \delta]$;

(B) $\bar{x} \in [-\delta, 0],\ \bar{y} \in [0, \delta],\ \bar{x} + \bar{y} \in [-\delta, 0]$;

(C) $\bar{x} \in [0, \delta],\ \bar{y} \in [0, \delta],\ \bar{x} + \bar{y} \in [0, \delta]$;

(D) $\bar{x} \in [-\delta, 0],\ \bar{y} \in [-\delta, 0],\ \bar{x} + \bar{y} \in [-\delta, 0]$;
(E) $\bar{x} \in [0, \delta],\ \bar{y} \in [-\delta, 0],\ \bar{x} + \bar{y} \in [0, \delta]$;
(F) $\bar{x} \in [0, \delta],\ \bar{y} \in [-\delta, 0],\ \bar{x} + \bar{y} \in [-\delta, 0]$;

Case (A) is trivial.

In case (B) we get the following inequalities from (38):

– $|f(\bar{y}) - f(\bar{x} + \bar{y}) - f(-\bar{x})| \le K$, with $x = \bar{x} + \bar{y}$ and $y = -\bar{x}$;
– $|-f(0) + f(\bar{x}) + f(-\bar{x})| \le K$, with $x = \bar{x}$ and $y = -\bar{x}$;
– $|f(\bar{y}) - f(0) - f(\bar{y})| \le K$, with $x = 0$ and $y = \bar{y}$;

and the addition of these inequalities implies (39).

In case (F) we get (39) by case (B) and with $x = \bar{y}$ and $y = \bar{x}$.

The remaining cases can be treated by the substitutions
$x = -\bar{y}$ and $y = \bar{y}$; $x = -\bar{y}$ and $y = \bar{x} + \bar{y}$; $x = 0$ and $y = \bar{y}$ in case (C);
$x = \bar{y}$ and $y = -\bar{y}$; $x = \bar{x}$ and $y = -\bar{x} - \bar{y}$; $x = \bar{x} + \bar{y}$ and $y = -\bar{x} - \bar{y}$ in case (D); $x = \bar{y}$ and $y = \bar{x}$ in case (E), respectively.

Theorem 5. *Let $\alpha \ge 0$ $\alpha \ne 1$ be a real number and let $f : \mathbb{R} \to \mathbb{R}$ be a function with the property*

$$f(x+y) - f(x) - f(y) = o(y^\alpha) \quad (x \le 0 \le x+y,\ y \searrow 0). \tag{40}$$

Then there exists an additive function $a : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) - a(x) = o(|x|^\alpha) \quad (x \to 0).$$

Proof. For $\alpha > 1$ the statement is proved in Theorem 4.

In the sequel, $\alpha \in [0, 1)$. In this case the proof is similar to some reasoning in [7].

By (40) there exist real numbers $\delta > 0$ and $K > 0$ such that

$$|f(x+y) - f(x) - f(y)| \le K \quad (x \in [-\delta, 0],\ y, x+y \in [0, \delta]),$$

hence from Lemma 2 we have

$$|f(x+y) - f(x) - f(y)| \le 3K \quad (x, y, x+y \in [-\delta, \delta]).$$

Z. Kominek proved ([4], Lemma 1) that this property implies the existence of an additive function $a : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - a(x)| \le 12K \quad (x \in [-\delta, \delta]).$$

For the function $\varepsilon : [-\delta, \delta] \to \mathbb{R}$, $\varepsilon(x) = f(x) - a(x)$ we have $\varepsilon(0) = 0$ and by Theorem 1

$$\varepsilon(2z) - 2\varepsilon(z) = o(|z|^\alpha) \quad (z \to 0).$$

It is easy to see, that this property is equivalent to the following: there exist a real number $\delta_1 > 0$ and a continuous, increasing function $h : [0, \delta_1] \to \mathbb{R}$ such that $\lim_{z \searrow 0} h(z) = 0$ and

$$|\varepsilon(2z) - 2\varepsilon(z)| \le |z|^\alpha h(|z|) \quad (z \in [-\delta_1, \delta_1]).$$

Introducing the function

$$\bar{\varepsilon}(z) = \begin{cases} \dfrac{\varepsilon(z)}{|z|^\alpha}, & \text{if } z \in [-\delta_1, \delta_1],\ z \ne 0, \\ 0, & \text{if } z = 0 \end{cases}$$

we have

$$\left|\, |z|^\alpha \bar{\varepsilon}(z) - \frac{1}{2} 2^\alpha |z|^\alpha \bar{\varepsilon}(2z) \,\right| \le \frac{1}{2} |z|^\alpha h(|z|) \quad (z \in [-\delta_1, \delta_1])$$

and

$$|\bar{\varepsilon}(z) - 2^{\alpha-1} \bar{\varepsilon}(2z)| \le \frac{1}{2} h(|z|) \quad (z \in [-\delta_1, \delta_1]).$$

Write

$$s_k = \sup\left\{ |\bar{\varepsilon}(z)| \;\middle|\; \frac{\delta_1}{2^k} \le |z| \le \frac{\delta_1}{2^{k-1}} \right\} \quad (k \in \mathbb{N}).$$

Then

$$s_{k+1} \le 2^{\alpha-1} s_k + \frac{1}{2} h\left(\frac{\delta_1}{2^k}\right), \quad (k \in \mathbb{N})$$

therefore, $\lim_{k \to \infty} s_k = 0$ and

$$\varepsilon(z) = o(|z|^\alpha) \quad (z \to 0).$$

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