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A note on positive integer solutions of the equation $xy + yz + zx = n$

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Abstract. Let n be a positive integer. In this note we prove that if $n > 10^{13}$, then there exist at most seven exceptional values n for which the equation $xy+yz+zx=n$ has no positive integers (x, y, z). Moreover, under the assumption of the generalized Riemann conjecture, the above-mentioned exceptional values do not exist.

1. Introduction

Let $\mathbb{Z}$, $\mathbb{N}$ be the sets of integers and positive integers respectively. For a fixed positive integer n, let $S(n)$ denote the number of solutions (x, y, z) of the equation

$$xy + yz + zx = n, \quad x, y, z \in \mathbb{N}, \quad x \leq y \leq z. \tag{1}$$

In [6], Kovács examined that if $n \leq 10^7$, then $S(n) > 0$ except $n = 1$, 2, 4, 22, 30, 42, 58, 70, 78, 102, 130, 190, 210, 330 and 462. Let $E(X)$ denote the number of $n \leq X$ for which $S(n) = 0$. Cai [1] proved that $E(X) = O(X \cdot 2^{-(1-\varepsilon)(\log X)/\log\log X})$ for any $\varepsilon > 0$. Recently, Hassan, Brindza and Pintér [2] showed that if $S(n) = 0$, then the squarefree part of n belongs to a finite set which can be effectively determined up to at most one element. While $S(n) = 0$, n is called an exceptional value. In this note we prove the following result.

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Theorem. *If $n > 10^{13}$, then there exist at most seven exceptional values n. Moreover, under the assumption of the generalized Riemann conjecture, the above-mentioned exceptional values do not exist.*

2. Preliminaries

Lemma 1. *If $n > 1$ and $2 \nmid n$ or $n > 4$ and $4 \mid n$, then $S(n) > 0$.*

Proof. If $n > 1$ and $2 \nmid n$, then (1) has a solution $(x, y, z) = (1, 1, (n-1)/2)$. If $n > 4$ and $4 \mid n$, then (1) has a solution $(x, y, z) = (2, 2, n/4 - 1)$. The lemma is proved.

Lemma 2 ([7])**.** *For any positive integer t, let p_t denote the t-th odd prime. Then $p_t \geq \max(3, (t+1)\log(t+1))$.*

Lemma 3. *For any positive integer k,*

$$p_1 p_2 \cdots p_k > \frac{5}{2}\left(\frac{k+1}{e}\right)^{k+3/2} \prod_{t=3}^{k+1} \log t.$$

Proof. Using Lemma 2, we get

$$p_1 p_2 \cdots p_k > \frac{3}{2}(k+1)\,!\,(\log 3) \prod_{t=3}^{k+1} \log t. \tag{2}$$

By Stirling's theorem, we have

$$(k+1)\,! > \sqrt{2\pi(k+1)}\left(\frac{k+1}{e}\right)^{k+1}. \tag{3}$$

Substituting (3) into (2), we obtain the lemma immediately.

Lemma 4. *For any positive integer k, let $P(k) = \sqrt{8p_1p_2\cdots p_k} \,/\, \log 8p_1p_2\cdots p_k$. If $k \geq 5$, then $P(k+1) > 2P(k)$.*

Proof. Since p_i $(i = 1, 2, \ldots, k)$ are all odd primes for which do not exceed p_k, every prime factor q of $8p_1p_2\cdots p_k - 1$ satisfies $q \geq p_{k+1}$. It implies that $8p_1p_2\cdots p_k - 1 \geq p_{k+1}$. Hence, if $k \geq 5$, then we have

$$\frac{P(k+1)}{P(k)} = \sqrt{p_{k+1}}\,\frac{\log 8p_1p_2\cdots p_k}{\log 8p_1p_2\cdots p_k + \log p_{k+1}} > \frac{1}{2}\sqrt{p_{k+1}} \geq \frac{1}{2}\sqrt{17} > 2.$$

The lemma is proved.

We now recall some basic properties on class numbers of binary quadratic forms (see [3]). For any positive integer m, let $h(m)$ and $h_0(m)$ denote the class number of binary quadratic forms and binary quadratic primitive forms with discriminant $-m$, respectively. Then we have

$$h(m) = \sum_{d^2 \mid m} h_0\left(\frac{m}{d^2}\right), \tag{4}$$

where d^2 runs through all square divisors of m. If $m \equiv 0$ or $3 \pmod 4$, then the negative discriminant $-m$ can be written as

$$-m = -fg^2, \tag{5}$$

where $-f$ is a fundamental discriminant, g is a positive integer. Further, we have

$$h_0(m) = h_0(fg^2) = h_0(f) \prod_{p \mid g} \left(1 - \left(\frac{-f}{p}\right) \frac{1}{p}\right) \tag{6}$$

and

$$h_0(f) = \frac{\sqrt{f}}{\pi} L(1, \chi), \quad \text{if } f > 4, \tag{7}$$

where p runs through all prime factors of g, $(-f/p)$ is Kronecker's symbol, χ is a real primitive character modulo f, $L(s, \chi)$ is the Dirichlet L-function associated with χ. Furthermore, let $\omega(f)$ denote the number of distinct prime factors of f, then we have

$$2^{\omega(f)-1} \mid h_0(f). \tag{8}$$

Lemma 5 ([4, 5])**.** *Let χ be a real primitive character modulo q. For any positive number ε with $\varepsilon \le 0.0723$, if $q > e^{1/\varepsilon}$, then*

$$L(1, \chi) > \min\left(\frac{1}{7.735 \log q}, \frac{2.865\varepsilon}{q^{\varepsilon}}\right),$$

except at most one exceptional modulo q. Moreover, under the assumption of the generalized Riemann conjecture, the exceptional modulo does not exist.

3. Proof of Theorem

Let n be an exceptional value with $n > 10^{13}$. By Lemma 1, we have $2 \parallel n$. Then n can be written in the form

$$n = 2\ell_1\ell_2\cdots\ell_k\, u_1^{2\alpha_1}u_2^{2\alpha_2}\cdots u_r^{2\alpha_r}\, v_1^{2\beta_1+1}v_2^{2\beta_2+1}\cdots v_s^{2\beta_s+1}, \tag{9}$$

where $\ell_1,\ell_2,\ldots,\ell_k,\ u_1,u_2,\ldots,u_r,\ v_1,v_2,\ldots,v_s$ are distinct odd primes, $\alpha_1,\alpha_2,\ldots,\alpha_r,\ \beta_1,\beta_2,\ldots,\beta_s$ are positive integers. We may assume that $\ell_1 < \ell_2 < \ldots < \ell_k$, $u_1 < u_2 < \ldots < u_r$ and $v_1 < v_2 < \ldots < v_s$. Notice that $3!\,S(n) \geq 3h(4n) - 3d(n)$ by [2, page 201], where $d(n)$ is the number of positive divisors of n. Then we have

$$d(n) \geq h(4n). \tag{10}$$

By (5) and (9), we have

$$-4n = -fg^2, \tag{11}$$

where

$$f = 8\ell_1\ell_2\cdots\ell_k v_1v_2\cdots v_s, \quad g = u_1^{\alpha_1}u_2^{\alpha_2}\cdots\ u_r^{\alpha_r}\ v_1^{\beta_1}v_2^{\beta_2}\cdots v_s^{\beta_s}. \tag{12}$$

and $-f$ is a fundamental discriminant. Further, by (4), (6), (11) and (12), we get

$$\begin{aligned} h(4n) &= \sum_{d^2|4n} h_0\left(\frac{4n}{d^2}\right) = \sum_{d^2|g^2} h_0\left(f\frac{g^2}{d^2}\right) \\ &= \sum_{d^2|g^2} h_0(f)\left(\prod_{p|g/d}\left(1-\left(\frac{-f}{p}\right)\frac{1}{p}\right)\right) \\ &= h_0(f)\sum_{d|g}\left(\prod_{p|g/d}\left(1-\left(\frac{-f}{p}\right)\frac{1}{p}\right)\right). \end{aligned} \tag{13}$$

Since $(-f/p) \leq 1$, we deduce from (12) and (13) that

$$\begin{aligned} h(4n) &\geq h_0(f)\sum_{d|g}\left(\prod_{p|g/d}\left(1-\frac{1}{p}\right)\right) \\ &= h_0(f)\left(\prod_{i=1}^{r}(u_i^{\alpha_i}-1)\right)\left(\prod_{j=1}^{s}\left(v_j^{\beta_j}-1\right)\right). \end{aligned} \tag{14}$$

On the other hand, by (9), we have

$$d(n) = 2^{k+1}\left(\prod_{i=1}^{r}(2\alpha_i+1)\right)\left(\prod_{j=1}^{s}(2\beta_j+2)\right). \tag{15}$$

The combination of (10), (14) and (15) yields

$$2^{k+s+1} \geq h_0(f)\left(\prod_{i=1}^{r}\frac{u_i^{\alpha_i}-1}{2\alpha_i+1}\right)\left(\prod_{j=1}^{s}\frac{v_j^{\beta_j}-1}{\beta_j+1}\right). \tag{16}$$

We first consider the case of $g = 1$. Then we have $r = s = 0$, and by (16), we get

$$2^{k+1} \geq h_0(f). \tag{17}$$

Using Lemma 5, by (7), we have

$$h_0(f) = \frac{\sqrt{f}}{\pi}\,L(1,\chi) > \begin{cases} \dfrac{\sqrt{f}}{7.735\pi\log f}, & \text{if } 4\cdot 10^{13} < f \leq 10^{28}, \\ \dfrac{0.2071 f^{0.4277}}{\pi}, & \text{if } f > 10^{28}, \end{cases} \tag{18}$$

except at most one exceptional value. Let p_t denote the t-th odd prime. We see from (12) that $f \geq 8p_1p_2\cdots p_k$. Therefore, by Lemma 3, we get

$$f > 20\left(\frac{k+1}{e}\right)^{k+3/2}\prod_{t=3}^{k+1}\log t. \tag{19}$$

Substituting (19) into (18), if $f > 10^{28}$, then from (17) and (18) we get

$$654\,(12.77)^k > (k+1)^{k+3/2}\prod_{t=3}^{k+1}\log t. \tag{20}$$

We calculate from (20) that $k \leq 10$. Since $f > 10^{28}$, by (17) and (18), we get $8\cdot 10^3 > 2^{k+1}\,\pi > 0.2071 f^{0.4277} > 10^{10}$, a contradiction.

If $4\cdot 10^{13} < f \leq 10^{28}$, then from (17) and (18) we get

$$2^{k+1} > \frac{\sqrt{f}}{7.735\pi\log f}. \tag{21}$$

By Lemma 4, if $k \geq 11$, then from (21) we get

$$\begin{aligned} 10^5 \cdot 2^{k-11} > 2^{k+1} \cdot 7.735\,\pi > \frac{\sqrt{f}}{\log f} \geq \frac{\sqrt{8p_1p_2\cdots p_k}}{\log 8p_1p_2\cdots p_k} \\ > 2^{k-11}\frac{\sqrt{8p_1p_2\cdots p_{11}}}{\log 8p_1p_2\cdots p_{11}} > 1.6\cdot 10^5 \cdot 2^{k-11}, \end{aligned} \tag{22}$$

a contradiction. So we have $k \leq 10$, and by (21),

$$49767 > \frac{\sqrt{f}}{\log f}. \tag{23}$$

Since $f > 4 \cdot 10^{13}$, (23) is impossible. Thus, there exists at most one exceptional value n such that $n > 10^{13}$ and $g = 1$. Moreover, by Lemma 5, under the assumption of the generalized Riemann conjecture, the above-mentioned exceptional value does not exist.

We next consider the case of $g > 1$. By (8) and (12), we have $h_0(f) \geq 2^{k+s}$. Hence, by (16), we obtain

$$2 \geq \left(\prod_{i=1}^{r} \frac{u_i^{\alpha_i}-1}{2\alpha_i+1}\right)\left(\prod_{j=1}^{s} \frac{v_j^{\beta_j}-1}{\beta_j+1}\right). \tag{24}$$

Recall that $u_1, u_2, \ldots, u_r$, $v_1, v_2, \ldots, v_s$ are distinct odd primes satisfying $u_1 < u_2 < \cdots < u_r$ and $v_1 < v_2 < \cdots < v_s$. We find from (24) that

$$\begin{aligned} & r=1, \quad s=0, \quad (u_1,\alpha_1) = (3,1),(3,2),(5,1),(7,1); \\ & r=2, \quad s=0, \quad (u_1,u_2,\alpha_1,\alpha_2) = (3,5,1,1),(3,7,1,1); \\ & r=0, \quad s=1, \quad (v_1,\beta_1) = (3,1),(5,1); \\ & r=1, \quad s=1, \quad (u_1,v_1,\alpha_1,\beta_1) \\ & \qquad = (3,5,1,1),(3,7,1,1),(5,3,1,1),(7,3,1,1). \end{aligned} \tag{25}$$

From (12) and (25), we obtain

$$g \in \{3,5,7,9,15,21\}. \tag{26}$$

Notice that

$$\left(\prod_{i=1}^{r} \frac{u_i^{\alpha_i}-1}{2\alpha_i+1}\right)\left(\prod_{j=1}^{s} \frac{v_j^{\beta_j}-1}{\beta_j+1}\right) > \frac{2}{3}. \tag{27}$$

We get from (16) and (27) that

$$2^{k+s} \cdot 3 \geq h_0(f). \tag{28}$$

Using the same method as in the case of $g = 1$, we can prove that there exists at most one f for which (28) holds. Moreover, under the assumption of the generalized Riemann conjecture, such f does not exist. Thus, by (26), there exist at most six exceptional values n with $g > 1$. To sum up, the proof is complete.

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