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Monotone vector fields

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Abstract. Monotone vector fields on Riemannian manifolds will be introduced. Their first order characterizations will be given. The connection with one parameter transformation groups, the Lie derivative and conformal vector fields will be outlined.

1. Introduction

We define the notion of geodesic monotone vector fields on Riemannian manifolds. This notion depends on the metrical tensor of the manifold. In particular, if we consider a finite dimensional Banach space B with the scalar product induced by the natural pairing $\langle\, .\, ,\, .\,\rangle : B \times B^* \to \mathbb{R}$ through the identification of B with its dual B^*, then the geodesic monotone vector fields on B can be identified with the monotone operators on B (in the sense of Minty–Browder [2], [11]). More precisely $A : B \to B$ is monotone if and only if the vector field $x \mapsto (x, Ax)$ is monotone.

C. Udrişte proves [9] that a function f defined on an open and geodesic convex subset of a Riemannian manifold is geodesic convex, if and only if its gradient is geodesic monotone. This result is stated and is expressed in terms of the differential of f, in [10] too. T. Rapcsák in [8] reformulates the result of Udrişte, giving explicitly the inequality which expreses the geodesic monotonicity of the gradient of a geodesic

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convex function. However, as far as we know, nobody has been talking (in general) about geodesic monotone vector fields yet.

We can generalize the notion of scalar derivative (see [3], [6]), introduced by us, for characterizing such vector fields. The results are generalizations of some results of [3] and [6].

The theory of one parameter transformation groups is useful tool for many application oriented investigations. In this paper we connect the topic of one parameter transformation groups with results of nonlinear analysis on Riemannian manifolds, concerning geodesic monotonicity. These results generalize the results of [5]

We relate the geodesic monotone vector fields and the expansive maps. (A map $\phi : M \to M$ is called expansive if its tangential maps are expansive in every point of M.) This is done through the Killing monotone vector fields, a notion introduced by us. The Killing monotone vector fields are vector fields on a Riemannian manifold (M, g), generated by expansive one parameter transformation groups, where we mean by expansive one parameter transformation groups smooth one parameter transformation groups ϕ_t over Riemannian manifolds, with ϕ_t expansive for $t > 0$ (from this follows easily that ϕ_t is nonexpansive for $t < 0$). The Killing strictly monotone vector fields can be introduced similarly. We prove that a vector field X on M is Killing monotone (Killing strictly monotone) if and only if (if) the Lie derivative $\mathcal{L}_X g$, of the metrical tensor g with respect to X is positively semidefinite (positively definite) in every point of M. The positive semidefinitness (positive definiteness) of $\mathcal{L}_X g$ is proved to be equivalent with the positive semidefiniteness (positive definiteness) of the endomorphism A_X, defined by $A_X U = \nabla_U X$, with respect to g, where ∇ is the Levi–Civitá connection of M.

We prove that a vector field is Killing monotone if and only if it is geodesic monotone. From these follows that a necessary and sufficient condition for X to be geodesic monotone is the positive semidefiniteness of $\mathcal{L}_X g$ in every point.

We express the lower geodesic scalar derivative of X in terms of the Lie derivative, and prove that a necessary condition for X to be strictly geodesic monotone is the positive definiteness of $\mathcal{L}_X g$ in every point of M.

Finally we prove that X is geodesic scalar differentiable if and only if it is conformal.

Although the results of this paper are theoretical, they could be the starting point for solving new optimization problems, since they are connected to geodesic convexity (see [7] and [8]).

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2. Geodesic monotone vector fields

Definition 2.1. Let M be a Riemannian manifold.

(i) A subset K of M is called *geodesic convex* [7] if for every two points of M there is a geodesic arc contained in K joining these points.

(ii) Let K be a geodesic convex subset of M. A function $f : K \to \mathbb{R}$ is called geodesic convex (strictly geodesic convex), [7] if $f \circ \gamma : [0, l] \to \mathbb{R}$ is convex (strictly convex) for every unit speed geodesic arc $\gamma : [0, l] \to M$ contained in K.

If N is an arbitrary manifold, we shall denote by $\mathrm{Sec}(TN)$ the family of sections of the tangent bundle TN of N. Using this notation we have the following definition:

Definition 2.2. Let (M, g) be a Riemannian manifold, $K \subset M$ a geodesic convex open set and $X \in \mathrm{Sec}(TK)$ a vector field on K.

X is called geodesic *monotone* if for every $x, y \in K$ and every unit speed geodesic arc $\gamma : [0, l] \to M$ joining x and y ($\gamma(0) = x$, $\gamma(l) = y$) and contained in K we have that

$$g(X_x, \dot\gamma(0)) \le g(X_y, \dot\gamma(l)),$$

where $\dot\gamma$ denotes the tangent vector of γ with respect to the arclength.

Let X be geodesic monotone. With the previous notations we have:

X is called *strictly geodesic monotone* if for every distinct x and y we have

$$g(X_x, \dot\gamma(0)) < g(X_y, \dot\gamma(l)).$$

X is called *virtually geodesic monotone* if there are x and y such that

$$g(X_x, \dot\gamma(0)) < g(X_y, \dot\gamma(l)).$$

X is called *trivially geodesic monotone* if for every x, y we have that

$$g(X_x, \dot{\gamma}(0)) = g(X_y, \dot{\gamma}(l)).$$

Since the length of the tangent vector of an arbitrary parametrized geodesic is constant, the relations of Definition 2.2 can be given for any parametrization of γ.

It is also easy to see that X is geodesic monotone (strictly geodesic monotone), if and only if for every arbitrarily parametrized geodesic γ the function

$$\upsilon : \tau \mapsto g(X_{\gamma(\tau)}, \gamma'(\tau))$$

is monotone (strictly monotone), where $\gamma'(\tau)$ is the tangent vector of γ, with respect to its parameter τ. Similarly X is trivially monotone, if and only if υ is constant.

Definition 2.2 can be given for an arbitrary open subset G of M, if we consider just a pair of points x, y which can be joined by a geodesic contained in G. Hence we can define the (global) monotonicity notion on every manifold, not just on the complete ones.

The following theorem is a modified version of Udrişte's result [10], [9].

Theorem 2.3. *Let M be a Riemannian manifold and K an open and geodesic convex subset of M. A function $f : K \to \mathbb{R}$ is geodesic convex (strictly geodesic convex), if and only if its gradient* grad f *is geodesic monotone (strictly geodesic monotone).* (Udrişte, 1976)

C. Udrişte gives the inequality which expreses the geodesic monotonicity in terms of df, the differential of f.

T. Rapcsák [8] states the required inequality in explicit form, using the gradient of f. However, neither of them speaks (in general) about monotone vector fields.

Definition 2.4. Let (M, g) be a Riemannian manifold, $K \in M$ a geodesic convex open set and $X \in \mathrm{Sec}(TK)$ a vector field on K. Then the *lower geodesic scalar derivative* (*upper geodesic scalar derivative*) of X is the function

$$\underline{X}^{\#} : K \to \mathbb{R};\ \underline{X}^{\#}(x) = \liminf_{\substack{t \searrow 0 \\ \gamma \in \Gamma}} \frac{g(X_{\gamma(t)}, \dot{\gamma}(t)) - g(X_x, \dot{\gamma}(0))}{t}$$

$$\left(\overline{X}^{\#} : K \to \mathbb{R};\ \overline{X}^{\#}(x) = \limsup_{\substack{t \searrow 0 \\ \gamma \in \Gamma}} \frac{g(X_{\gamma(t)},\, \dot{\gamma}(t)) - g(X_x,\, \dot{\gamma}(0))}{t} \right),$$

where Γ denotes the family of unit speed geodesic arcs $\gamma : [0,l] \to M$ starting from x (i.e. $\gamma(0) = x$), and contained in K.

If $x_0 \in K$ and $\underline{X}^{\#}(x_0) = \overline{X}^{\#}(x_0) =: X^{\#}(x_0)$, then X is called *geodesic scalar differentiable in* x_0 and $X^{\#}(x_0)$ is called the *geodesic scalar derivative of* X *in* x_0. If X is geodesic scalar differentiable in every $x \in K$ then X is called *geodesic scalar differentiable*, and the function $X^{\#} : x \mapsto X^{\#}(x)$ is called the *geodesic scalar derivative* of X.

The following theorem is a local characterization of geodesic monotone vector fields.

Theorem 2.5. *Let (M, g) be a Riemannian manifold, $K \subset M$ a geodesic convex open set and $X \in \mathrm{Sec}(TK)$ a vector field on K. Then we have the following assertions:*

(i) *X $(-X)$ is geodesic monotone if and only if $\underline{X}^{\#}(x) \geq 0$ $(\overline{X}^{\#}(x) \leq 0)$ for all $x \in K$.*

(ii) *X is trivially monotone if and only if X is scalar differentiable and $X^{\#}(x) = 0$ for all $x \in K$*

Proof. (i) First we suppose that $\underline{X}^{\#}(x) \geq 0$ for every $x \in K$ and prove that X is geodesic monotone Let a and b be two arbitrary distinct points of K and $\gamma : [0,l] \to M$ a unit speed geodesic arc joining a and b $(\gamma(0) = a$, $\gamma(l) = b)$ contained in K. Let $x = \gamma(t)$ be an arbitrary point of the geodesic arc.

Let $\epsilon > 0$ be an arbitrary but fixed positive number. Since $\underline{X}^{\#} \geq 0$, there is a $\delta(t) > 0$ such that for every $s \in I_t =]t - \delta(t), t + \delta(t)[$ we have that

$$\frac{g(X_{\gamma(s)}, \dot{\gamma}(s)) - g(X_x, \dot{\gamma}(t))}{s - t} > -\frac{\epsilon}{l}. \tag{2.1}$$

(The geodesic arc can be continued to an open one contained in K and $\delta(t)$ chosen sufficiently small so that $\gamma(s)$ can be defined.) But $\{I_t : t \in [0,l]\}$ is an open covering of the compact set $[0,l]$. Hence $[0,l] \subset I_{t_1} \cup I_{t_2} \cup \ldots \cup I_{t_{m-1}}$ for some positive integer m and some points $t_1 < t_2 < \ldots < t_{m-1}$ of $[0,l]$. This yields $0 =: t_0 \in I_{t_1}$ and $l =: t_m \in I_{t_{m-1}}$. Obviously we can choose the intervals $\{I_{t_i} : i = \overline{1, m-1}\}$ so that no interval is contained in another. Let $\xi_i \in I_{t_{i-1}} \cap I_{t_i} \cap]t_{i-1}, t_i[$ for $i = \overline{1,m}$. Then, using (2.1) we have that

$$g(X_{\gamma(\xi_i)}, \dot{\gamma}(\xi_i)) - g(X_{\gamma(t_{i-1})}, \dot{\gamma}(t_{i-1})) > -\frac{\epsilon}{l}(\xi_i - t_{i-1}), \tag{2.2}$$

and

$$g(X_{\gamma(t_i)}, \dot\gamma(t_i)) - g(X_{\gamma(\xi_i)}, \dot\gamma(\xi_i)) > -\frac{\epsilon}{l}(t_i - \xi_i), \tag{2.3}$$

for $i = \overline{1, m}$. Summing the inequalities (2.2) and (2.3) we obtain

$$g(X_{\gamma(t_i)}, \dot\gamma(t_i)) - g(X_{\gamma(t_{i-1})}, \dot\gamma(t_{i-1})) > -\frac{\epsilon}{l}(t_i - t_{i-1}), \tag{2.4}$$

for $i = \overline{1, m}$. Summing the inequalities (2.4) for $i = \overline{1, m}$ we get

$$g(X_{\gamma(l)}, \dot\gamma(l)) - g(X_{\gamma(0)}, \dot\gamma(0)) > -\epsilon.$$

Since ϵ is an arbitrary positive number we have that

$$g(X_{\gamma(l)}, \dot\gamma(l)) - g(X_{\gamma(0)}, \dot\gamma(0)) \geq 0.$$

Hence for any two distinct points a and b of K and an arbitrary geodesic arc joining a and b contained in K, we have that

$$g(X_b, \dot\gamma(l)) \geq g(X_a, \dot\gamma(0)).$$

Thus X is geodesic monotone.

Now suppose that X is geodesic monotone. Then by the definitions of geodesic monotonicity and geodesic scalar derivative, we have that $\underline{X}^{\#}(x) \geq 0$ for all $x \in K$. The statement of the theorem for $-X$ is obtained by using the identity

$$\underline{(-X)}^{\#} = -\overline{X}^{\#}.$$

(ii) We have that X is trivially geodesic monotone if and only if X and $-X$ is geodesic monotone. Hence X is trivially geodesic monotone if and only if $0 \leq \underline{X}^{\#}(x) \leq \overline{X}^{\#}(x) \leq 0$ for all $x \in K$. Hence $\underline{X}^{\#}(x) = \overline{X}^{\#}(x) = 0$. Thus X is trivially geodesic monotone if and only if it is scalar differentiable and $X^{\#}(x) = 0$ for every $x \in K$. □

Similarly to Theorem 2.5 we can prove the following

Theorem 2.6. *Let (M, g) be a Riemannian manifold, $K \subset M$ a geodesic convex open set and $X \in \mathrm{Sec}(TK)$ a vector field on K. If $\underline{X}^{\#}(x) > 0$ for every $x \in K$, then X is strictly geodesic monotone.*

The converse of this theorem is not true. As an example, it is enough to take the vector field C on $\mathbb{R}$ which assigns to each $x \in \mathbb{R}$ the tangent vector at 0 of the curve $t \mapsto x + tx^3$.

For the smooth case we have the following corollary of Theorem 2.5:

Corollary 2.7. *Let (M, g) be a Riemannian manifold, ∇ the Levi–Civitá connection of M, $K \subset M$ a geodesic convex set and $X \in \mathrm{Sec}(TK)$ a smooth vector field on K. For $x \in K$ define $\phi(x) : T_x(M) \to T_x(M)$ by $\phi(x)(v) = \nabla_v X$ for $v \in T_x(M)$, where $T_x(M)$ is the tangent space of M in x. Then we have the following assertions:*

(i) $\underline{X}^{\#}(x) = \inf\limits_{\substack{v \in T_x(M)\\ \|v\|=1}} g(\phi(x)(v), v)$

(ii) *X is geodesic monotone if and only if $\phi(x)$ is positive semidefinite for every x in K,*

(iii) *X is trivially monotone if and only if $\phi(x)$ is antisymmetric for every $x \in K$,*

(iv) *If $\phi(x)$ is positive definite for every $x \in K$, then X is strictly geodesic monotone.*

Proof. Let x be an arbitrary point of K. Then we have

$$\underline{X}^{\#}(x) = \inf_{\gamma \in \Gamma} \lim_{t \searrow 0} \frac{g(X_{\gamma(t)}, \dot\gamma(t)) - g(X_x, \dot\gamma(0))}{t}. \tag{2.5}$$

Equation (2.5) can be rewritten as

$$\underline{X}^{\#}(x) = \inf_{\gamma \in \Gamma} \frac{d}{dt}\Big|_{t=0} g(X_{\gamma(t)}, \dot\gamma(t)). \tag{2.6}$$

Using that ∇ is the Levi–Civitá connection of M and γ is geodesic, (2.6) becomes

$$\underline{X}^{\#}(x) = \inf_{\gamma \in \Gamma} g(\nabla_{\dot\gamma(0)} X|_x, \dot\gamma(0)). \tag{2.7}$$

Since for every $v \in T_x(M)$ there is $\gamma \in \Gamma$ such that $\dot\gamma(0) = v$, from (2.7) we obtain that

$$\underline{X}^{\#}(x) = \inf_{\substack{v \in T_x(M)\\ \|v\|=1}} g(\phi(x)(v), v), \tag{2.8}$$

where $\| \,.\, \|$ is the norm generated by the metrical tensor of M. From (2.8) and Theorems 2.5, and 2.6 we obtain the assertions (ii), (iii) and (iv) of the corollary. □

Denote by $\mathcal{T}M \subset \mathrm{Sec}(TM)$ the set of all smooth vector fields on M. Using Theorem 2.5 and Corollary 2.7 we obtain:

Theorem 2.8. *Let (M, g) be a Riemannian manifold, $K \subset M$ a geodesic convex open set and $X \in \mathcal{T}K$ a smooth vector field on K. Then the following assertions are equivalent:*

(i) *X is trivially geodesic monotone,*

(ii) *X is a Killing vector field.*

Proof. By Corollary 2.7 (iii) we have that X is trivially geodesic monotone if and only if A_Y is antysimmetric, where A_Y is the endomorphism of smooth vector fields defined by $A_Y(X) = \nabla_Y X$. But this is exactly a necessary and sufficient condition for X to be a Killing vector field. $\square$

3. Killing monotone vector fields

Let (M, g) be a Riemannian manifold.

Definition 3.1. A smooth map $\phi : M \to M$ is called expansive (*nonexpansive*), if for all $x \in M$ the tangent map $T\phi(x) : T_xM \to T_xM$ of ϕ is expansive (nonexpansive) i.e.

$$\|T\phi(x)U\| \geq \|U\| \quad (\|T\phi(x)U\| \leq \|U\|) \tag{3.1}$$

for all $U \in T_xM$, where $\| \ . \ \|$ is the norm generated by the metric g.

ϕ is called strictly expansive (*nonexpansive*) if in (3.1) we have strict inequalities for $U \neq 0$.

Definition 3.2. A smooth one parameter transformation group $\phi_t : M \to M$ is called (*nonexpansive*) if ϕ_t is expansive (nonexpansive) for all $t > 0$. The *strictly expansive* (*strictly nonexpansive*) *one parameter transformation groups* can be defined similarly.

Definition 3.3. Let $\phi_t : M \to M$ an expansive (nonexpansive) one parameter transformation group and $X : M \to TM$ the vector field generating ϕ_t i.e.

$$X(z) = \frac{d}{dt}\Big|_{t=0} \phi_t(z),$$

for all $z \in M$.

Then the vector field X is called an *infinitesimal expansion* (*nonexpansion*). The *infinitesimal strict expansions* (*infinitesimal strict nonexpansions*) can be defined similarly. The infinitesimal expansions (infinitesimal strict expansions) will also be called *Killing monotone vector fields* (*Killing strictly monotone vector fields*).

Remark 3.4. The Killing vector fields are trivially Killing monotone vector fields. Hence the Killing monotone vector fields are generalizations of the Killing vector fields.

The following theorem gives a characterization of the Killing monotone vector fields using the Lie derivative

Theorem 3.5. *$X : M \to TM$ is a Killing monotone vector field (Killing strictly monotone vector field) if and only if (if) the Lie derivative of the metrical tensor $\mathcal{L}_X g$ is positive semidefinite (positive definite) in every point of M.*

Proof. Suppose that X is Killing monotone. Let $U, V \in \mathcal{T}M$ be two arbitrary vector fields, $z \in M$ an arbitrary point and ϕ_t the one parameter transformation group generated by X. Then we have that

$$(\mathcal{L}_X g)(U,V) = X(g(U,V)) - g([X,U],V) - g(U,[X,\, V]). \tag{3.2}$$

If we put $V = U$ in (3.2) then we obtain

$$(\mathcal{L}_X g)(U,U) = X(g(U,U)) - 2g([X,U],U). \tag{3.3}$$

On the other hand

$$X(g(U,U))_z = \lim_{\substack{t\to 0\\ t>0}} \frac{g(U(\phi_t(z)),U(\phi_t(z))) - g(U(z),\, U(z))}{t}. \tag{3.4}$$

Since X is Killing monotone we have that

$$g(U(\phi_t(z)),\, U(\phi_t(z))) \geq g(T\phi_t^{-1}U(\phi_t(z)),\, T\phi_t^{-1}U(\phi_t(z))) \tag{3.5}$$

Inserting (3.5) in (3.4) we get

$$\begin{aligned} X(g(U,U))_z \geq \lim_{\substack{t\to 0\\ t>0}} \Bigg(& g\left(\frac{T\phi_t^{-1}U(\phi_t(z)) - U(z)}{t}, T\phi_t^{-1}U(\phi_t(z))\right) \\ & + g\left(U(z),\, \frac{T\phi_t^{-1}U(\phi_t(z)) - U(z)}{t}\right)\Bigg) \end{aligned} \tag{3.6}$$

But

$$\lim_{\substack{t\to 0\\ t>0}} \frac{T\phi_t^{-1}U(\phi_t(z)) - U(z)}{t} = [X,\, U]_z. \tag{3.7}$$

Inserting (3.7) in (3.6) we obtain

$$X(g(U,U))_z \geq 2g([X,U],U)_z. \tag{3.8}$$

From (3.3) and (3.8) it follows that $\mathcal{L}_X g$ is positive semidefinite at every $z \in M$.

Conversely, suppose that $\mathcal{L}_X g$ is positive semidefinite (positive definite) at every $z \in M$. We have to prove that $g(T\phi_t U, T\phi_t U) \geq g(U,U)$ $(g(T\phi_t U, T\phi_t U) > g(U,U))$ for all $t > 0$ and $U \in \mathcal{T}M$ $(U \in \mathcal{T}M \setminus \{0\})$. But this is trivial since

$$\frac{d}{dt} g(T\phi_t U, T\phi_t U)|_{t=0} = (\mathcal{L}_X g)(U, U). \qquad \square$$

Let $A \in \mathcal{T}M$ and let $A_X : \mathcal{T}M \to \mathcal{T}M$ be the endomorphism defined by

$$A_X U = \nabla_U X,$$

where ∇ is the Levi–Civitá connection of M. Then we have the following lemma:

Lemma 3.6.

$$\mathcal{L}_X g(U,U) = 2g(A_X U, U).$$

Proof. Since ∇ is the Levi–Civitá covariant derivative of M we have that

$$Xg(U,U) = 2g(\nabla_X U, U). \tag{3.9}$$

Inserting (3.9) in (3.3), bearing in mind that ∇ is torsion free and using the definition of A_X we obtain the required relation. $\square$

By Theorem 3.5 and Lemma 3.6 we have the following theorem:

Theorem 3.7. *$X \in \mathcal{T}M$ is Killing monotone (Killing strictly monotone) if and only if (if) A_X is positive semidefinite (positive definite) relative to the metrical tensor g.*

Using Theorem 3.7 and Corollary 2.7 it is easy to prove the following

Theorem 3.8. *Let $K \in M$ be an open and geodesic convex set and $X \in \mathcal{T}K$. Then we have the following two assertions:*

(i) *X is Killing monotone if and only if it is geodesic monotone.*

(ii) *X is a Killing vector field if and only if it is trivially geodesic monotone.*

Hence the notion of Killing monotone vector fields (Killing vector fields) coincides with that of geodesic monotone (trivially geodesic monotone) ones. We shall call for brevity' sake these vector fields monotone (trivially monotone). However, the relation between strictly geodesic monotone vector fields and Killing strictly monotone vector fields seems to be difficult and we cannot say anything about it yet.

From Theorem 3.7 and Corollary 2.7 (i) we get the following result:

Theorem 3.9. *Let $K \in M$ be an open and geodesic convex set and $X \in \mathcal{T}K$. Then X is Killing strictly monotone if $\underline{X}^{\#}(x) > 0$ for all $x \in K$.*

In [4] we gave examples of strictly geodesic monotone vector fields on the 3-dimensional half sphere and on the 3-dimensional hyperbolical space. These examples were given by proving the positiveness of the lower scalar derivative. Hence by Theorem 3.9 these vector fields are Killing strictly monotone too.

Using Lemma 3.6 and Corollary 2.7 (i), we easily get:

Theorem 3.10. *Let $X \in \mathcal{T}M$ be a smooth vector field on M and $x \in M$. Then the lower (upper) geodesic scalar derivative of X in x is given by the following formula:*

$$\underline{X}^{\#}(x) = \frac{1}{2} \inf_{g(h,h)=1} (\mathcal{L}_X g)_x(h,h) \quad \left(\overline{X}^{\#}(x) = \frac{1}{2} \sup_{g(h,h)=1} (\mathcal{L}_X g)_x(h,h)\right).$$

Theorem 3.11. *Using the previous notations, if $\mathcal{L}_X g$ is positive definite in every point of K then, X is strictly geodesic monotone.*

Proof. Let x be an arbitrary point of K. Since

$$\underline{X}^{\#}(x) = \frac{1}{2} \inf_{g(h,h)=1} (\mathcal{L}_X g)_x(h,h)$$

and the unit ball in T_xM is compact, there is a $h_0 \in T_xM$ with $g(h_0,h_0)=1$ such that

$$\underline{X}^{\#}(x) = \frac{1}{2}\mathcal{L}_X g_x(h_0,h_0) > 0.$$

Since x has been arbitrarily chosen Theorem 2.6 implies that X is strictly geodesic monotone. □

We recall the following definition:

Definition 3.12. Let (M, g) be a Riemannian manifold and X a smooth vector field on X. Denote by $\mathcal{L}_X g$ the Lie derivative of g with respect to X. Then X is called a *conformal vector field* if there is a smooth map $\lambda : M \to \mathbb{R}$ such that

$$\mathcal{L}_X g = \lambda g.$$

Theorem 3.13. *Let (M, g) be a Riemannian manifold and X a smooth vector field on X. Then X is geodesic scalar differentiable, if and only if X is conformal.*

Proof. By Theorem 3.10 X is geodesic scalar differentiable if and only if there is a smooth function $\lambda : M \to \mathbb{R}$ such that

$$\mathcal{L}_X g(Y, Y) = \lambda g(Y, Y) \tag{3.10}$$

for all $Y \in \mathcal{T}M$. From (3.2) it follows that $\mathcal{L}_X g$ is symmetric. But a symmetric bilinear form is determined by its corresponding quadratic form [1]. Hence (3.10) implies

$$\mathcal{L}_X g = \lambda g.$$

In addition we have that $\lambda = 2X^{\#}$. □

References

[1] I. M. Gelfand, Lectures on Linear Algebra, (Transl. from the 2nd Russian edition by A. Shenitzer), Reprint of the 1961 translation, publ. by *Interscience Publishers* (English), [2] Dover Books on Advanced Mathematics, *Dover Publications*, *New York*, 1989.

[2] G. Minty, Monotone operators in Hilbert spaces, *Duke Math. J.* **29** (1962), 341–346.

[3] S. Z. Németh, A scalar derivative for vector functions, *Rivista di Matematica Pura ed Applicata* no. 10 (1992), 7–24.

[4] S. Z. Németh, Geodesic monotone vector fields (*to appear*).

[5] S. Z. Németh, Scalar derivatives and conformity, *Annales Univ. Sci. Budapest.* **40** (1997), 105–111.

[6] S. Z. Németh, Scalar derivatives and spectral theory, *Mathematica, Thome* **35** (58) no. 1 (1993), 49–58.

[7] T. Rapcsák, Geodesic convexity in nonlinear optimization, *Journal of Optimization and Applications* **69** no. 1 (April 1991).

[8] T. Rapcsák, Smooth Nonlinear Optimization in $\mathbb{R}^n$, *Kluwer Academic Publishers*, 1997.

[9] C. Udrişte, Convex Functions on Riemannian Manifolds, *St. Cerc. Mat.* **28,** 6 (1976), 735–745.

[10] C. Udrişte, Continuity of Convex Functions on Riemannian Manifolds, *Bulletine Mathematique de Roumanie* **21** (1977), 215–218.

[11] E. Zeidler, Nonlinear Functional Analysis and its Applications, II/B: Nonlinear Monotone Operators, *Springer Verlag*, 1990.

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