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Euler products, Farey series and the Riemann hypothesis

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Dedicated to Professor Dr. Kálmán Győry on his sixtieth birthday

Abstract. For an integrable even function f with average value $\int_0^1 f(u)du = 0$ on the unit interval we consider the error term $E_f(x) = \sum_{\nu=1}^{\Phi(x)} f(\rho_\nu)$, where the summation extends over all Farey points of order $[x]$ through which we form the Mellin transform $F(s) = s\zeta(s)\int_1^\infty E_f(x)x^{-s-1}dx$ associated to f. We consider the equivalent assertions to the RH in terms of Farey series through some special choices of f.

In §2, for f given as a Fourier cosine series, we shall establish a Hecke-like correspondence between $f(\tau)$ and $F(s)$ with Ramanujan-like expansion of f with respect to the summatory functions of Ramanujan's function.

Then we shall go on to study a class of gap Fourier series. In Theorem 3 we shall consider the Weierstrass function $f(u)$ with a prime power gap (fractal). In Theorems 4 and 5 we shall consider a class of Fourier series f containing as a subclass, Riemann's and Takagi's function, respectively.

1. Introduction and statement of results

The purpose of the present paper is to examine the dynamical systematic aspect of the equivalence problem to the Riemann Hypothesis (RH) in terms of Farey series (first posed by Mikolás [8], [9]) as enunciated in Remark 3.2 of [7], thus flowing in a direction slightly different from,

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though in the same vein as, Parts I–IV [6], [15], [7] and [16]. (For the RH, cf. the paragraph containing (9) below.)

To proceed we define the Farey series $F_x = F_{[x]}$ of order $[x]$, $[x]$ denoting the integral part of x, to be the increasing sequence of irreducible fractions ρ_ν between 0 and 1 (0 exclusive) with denominator $\le x$. Then since the number of ρ_ν's with denominator n equals $\phi(n) = \sum_{k\le n,\ (k,n)=1} 1$, Euler's function, (k,n) denoting the g.c.d. of k and n, the total number $\# F_x$ of F equals

$$\Phi(x) = \sum_{n\le x} \phi(n),$$

the summatory function of Euler's function.

Now recall that if $f \in \mathcal{C}^1$, then the Euler–Maclaurin sum formula implies that

$$\sum_{\nu=1}^{\Phi(x)} f(\rho_\nu) = \Phi(x)\int_0^1 f(u)du + \frac{1}{2}(f(1)-f(0)) + \sum_{n\le x} M\left(\frac{x}{n}\right)\int_0^1 \overline{B}_1(nu)f'(u)du, \tag{1}$$

where

$$M(x) = \sum_{n\le x} \mu(n) \tag{2}$$

denotes the summatory function of the Möbius function $\mu(n)$, and $\overline{B}_1(u)$ denotes the 1st periodic Bernoulli polynomial given by the saw-tooth Fourier series

$$\overline{B}_1(u) = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2\pi nu}{n}$$

for $u \notin \mathbb{Z}$ and $\overline{B}_1(u) = 0$ for $u \in \mathbb{Z}$ ($\mathbb{Z}$ denotes the ring of integers).

Bearing (1) in mind, we define the error term $E_f(x)$ as in previous papers by

$$E_f(x) = \sum_{\nu=1}^{\Phi(x)} f(\rho_\nu) - \Phi(x)\int_0^1 f(u)du \tag{3}$$

associated to an integrable "core" function f. In some earlier papers we sometimes subtracted the second term $\frac{1}{2}(f(1)-f(0))$, which can be taken to be 0, since, on symmetry grounds, we may confine ourselves to even functions f only,

$$f(1-u)=f(u). \tag{4}$$

At this point giving an example is in order to get a general perspective.

Consider the discrete dynamical system $\{\varphi^n(u)\}$ on the unit interval (for dynamical systems, see e.g. Collett and Eckmann [1]) defined as the iterates of the tent function

$$\varphi(u)=\begin{cases} 2u & 0\le u\le \dfrac{1}{2} \\ 2-2u & \dfrac{1}{2}\le u\le 1, \end{cases}$$

$\varphi^1(u)=\varphi(u)$ and $\varphi^{n+1}(u)=\varphi(\varphi^n(u))$ $(n\in\mathbb{N})$.

Then it can be shown [18] that the RH is equivalent to

$$\sum_{\nu=1}^{\Phi(x)}\varphi^N(\rho_\nu)=\frac{1}{2}\Phi(x)+O\big(x^{\frac{1}{2}+\varepsilon}\big).$$

This follows from a more general result that the RH is equivalent to

$$\sum_{\nu=1}^{\Phi(x)}T_N(\rho_\nu)=\Phi(x)\int_0^1 T_N(u)du+O\big(x^{\frac{1}{2}+\varepsilon}\big),$$

where

$$T_N(u)=\sum_{n=1}^{N}\frac{1}{2^n}\varphi^n(u)$$

denotes the directly connected n tents of length $\frac{1}{2^n}$.

Since $T(u)=\lim_{N\to\infty}T_N(u)$ denotes the well-known Takagi function (ref. [13]) given by the gap (lacunary) Fourier series

$$T(u)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{2^m(2n+1)^2}\cos 2\pi 2^m(2n+1)u, \tag{5}$$

we are naturally led to consider the "infinite sum" of a discrete dynamical system.

In view of the remark on pp. 329–330 of [1] the Weierstrass function

$$f(u) = \sum_{n=-\infty}^{\infty} \cos 2\pi\beta^n u$$

is self-similar: $f(\beta u) = f(u)$ i.e. has a fractal figure as its graph.

We may say that in this paper we investigate the equivalence problem in terms of "attractors" of a class of discrete dynamical systems putting off the consideration of that side of affairs which is more directly related to dynamical systems till [18].

Turning back to a general situation we also note that we may assume f has its average value 0,

$$\int_0^1 f(u)du = 0 \tag{6}$$

by considering $f(u) - \int_0^1 f(u)du$ instead of f.

For the core function f given as a Fourier cosine series (which is the main object of our study)

$$f(u) = \sum_{n=0}^{\infty} c(n) \cos 2\pi nu,$$

average value 0 signifies that $c(0) = 0$, i.e.

$$f(u) = \sum_{n=1}^{\infty} c(n) \cos 2\pi nu. \tag{7}$$

On the assumption of (4) and (6), the error term reduces to

$$E_f(x) = \sum_{\nu=1}^{\Phi(x)} f(\rho_\nu) = \sum_{n \le x} M\left(\frac{x}{n}\right) \sum_{m \le n} f\left(\frac{m}{n}\right). \tag{8}$$

Recall that the RH to the effect that the Riemann zeta-function $\zeta(s)$ has no zeros on the critical line $\Re s = \sigma = \frac{1}{2}$ is equivalent to (cf. e.g. [3], [14])

$$M(x) = O\big(x^{\frac{1}{2}+\varepsilon}\big), \tag{9}$$

through which we study the equivalence problem.

As a main tool define the Mellin transform $F(s)$ associated to f:

$$F(s) = s\zeta(s)\int_1^\infty E_f(x)x^{-s-1}dx \tag{10}$$

for $\sigma > 1$, say.

The pair (f, F) (sometimes with subscripts) will always appear in this context throughout in what follows. Namely, f and F are in correspondence as a Mellin transform pair through $E_f(x)$.

In particular, if $E_f(x) = (M*a)(x) = \sum_{n\le x} M\left(\frac{x}{n}\right) a(n)$ with suitable $a(n)$, then $F(s)$ can be expressed as a Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}. \tag{11}$$

Conversely, if $F(s)$ is given by (11), then $E_f = M * a$, and f can be sometimes determined explicitly.

The principle which we apply to establishing equivalence is the following.

The Principle ([16], Lemma 1]). (i) If f is of Lipschitz class, then the RH implies

$$E_f(x) = (\text{Main term}) + O\big(x^{\frac{1}{2}+\varepsilon}\big)$$

or else we apply the Abelian theorem [17] to conclude the same.

(ii) Conversely, if $E_f(x) = O\big(x^{\frac{1}{2}+\varepsilon}\big)$ and $F(s)$ does not vanish for $\sigma > \frac{1}{2}$, then $\zeta(s)$ dose not vanish for $\sigma > \frac{1}{2}$, which is equivalent to the RH.

We shall make frequent use of the following.

Notation.

- Lipschitz space $\Lambda_\alpha = \{f : [0,1] \to \mathbb{C} \mid |f(u) - f(v)| < M|u-v|^\alpha$ for an absolute constant $M\}$.
- Ramanujan's sum $c_k(n) = \sum_{1\le h\le k,\ (h,k)=1} e^{2\pi i nh/k}$ to the modulus k.
- $\sigma_z(n) := \sum_{d|n} d^z$, the sum-of-divisors function.
- With $B_k(t)$ denoting the k-th Bernoulli polynomial, we let $\overline{B}_k(t) = B_k(\{t\}) = B_k(t-[t])$ be the k-th periodic Bernoulli polynomial.
- In Example 1 we shall use $S_s(x) = \sum\limits_{n\le x} M\left(\frac{x}{n}\right)$ and $\psi(x) = \sum\limits_{p^m\le x} \log p$, the von Mangoldt function (in particular, $S_{-1}(x) = \Phi(x)$, $S_0(x) = 1$).

2. Hecke-like correspondence and Euler product

In the special case where $f(u)$ is given as a Fourier (cosine) series, the pair (f, F) gives a *Hecke-like correspondence*, by which we mean an analogy between our pair (in which the argument u of f runs through $[0,1]$ only) and the Fourier expansion

$$f(\tau) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n\tau} \quad \tau \in \mathcal{H} \text{ (the upper half-plain)}$$

of a modular (cusp) form f and the associated Hecke L-function

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

(c.f. e.g. Hecke [2], Ogg [10]).

More specifically, we have

Theorem 1. *Suppose $f(u)$ has the Fourier expansion*

$$f(u) = \sum_{n=1}^{\infty} c(n) \cos 2\pi n u \tag{12}$$

satisfying the condition

$$\sum_{n=1}^{\infty} |c(n)| d(n) < \infty, \tag{13}$$

where $d(n)$ denotes the divisor function, then we have the Ramanujan-like expansion [12]

$$E_f(x) = \sum_{n=1}^{\infty} c(n) C_x(n), \tag{14}$$

where $C_x(n)$ denotes the summatory function $C_x(n) = \sum_{k \le x} c_k(n)$ of Ramanujan's sum, and with

$$a(n) = n \sum_{m=1}^{\infty} c(nm) \tag{15}$$

the generating Dirichlet series (11) *has the expansion*

$$F(s) = \sum_{n=1}^{\infty} \frac{1}{n^{s-1}} \sum_{m=1}^{\infty} c(nm) = \sum_{n=1}^{\infty} c(n)\sigma_{1-s}(n). \tag{16}$$

Conversely, if $F(s)$ *is given by* (11), *absolutely convergent for* $\sigma > 1$, *say then the Fourier coefficients of* f *are given by*

$$c(n) = \frac{1}{n} \sum_{k=1}^{\infty} \frac{\mu(k)}{k} a(kn). \tag{17}$$

Proof. By definition,

$$C_x(n) = \sum_{k \le x} \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \cos 2\pi n \frac{h}{k} = \sum_{\nu=1}^{\Phi(x)} \cos 2\pi n \rho_\nu = \sum_{m|n} M\left(\frac{x}{m}\right) m$$

by [15].

Hence

$$E_f(x) = \sum_{n=1}^{\infty} c(n) \sum_{m|n} M\left(\frac{x}{m}\right) m$$

which gives (14) after changing the order of summation.

Since

$$|E_f(x)| \le \sum_{n=1}^{\infty} |c(n)| \sum_{m|n} \frac{x}{m} m = x \sum_{n=1}^{\infty} |c(n)| d(n) = O(x)$$

by (13), this secures the problems of convergence, and

$$\int_1^{\infty} \left| \frac{E_f(x)}{x^{s+1}} \right| dx = O\left(\int_1^{\infty} x^{-\sigma} dx \right) = O(1) \quad \text{for } \sigma > 1,$$

whence it follows that $F(s)$ is absolutely convergent and (16) follows.

Conversely, by the Mellin inversion we deduce (17), where coefficients are given by (17).

We note that (17) is the Möbius inversion of (15) and hence that (15) and (17) are well-known Möbius inversion of infinite series. □

We may summarize this Heck-like correspondence schematically as follows.

$f(\tau) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n\tau}$	$\longleftrightarrow$	$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$
dynamical system generating Fourier series		generating Dirichlet series $\approx$ zeta-function with Euler product if a is multiplicative

Example 1 (See [6]).

$f(\tau)$	$F(s)$	$E_f(x)$
$\sum_{n=1}^{\infty} c(n) e^{2\pi i n\tau}$	$\sum_{n=1}^{\infty} c(n)\sigma_{1-s}(n) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$	$(M * a)(x)$
$\cos 2\pi\tau$	1	$M(x)$
$\sum_{n=1}^{\infty} e^{2\pi i n\tau} n^{-z} \quad (\Re z > 1)$	$\zeta(z)\zeta(s+z-1)$	$\zeta(z)S_{z-1}(x)$
$B_{2n}(\tau) \quad (n \in \mathbb{N})$	$B_{2n}\zeta(s+2n-1)$	$B_{2n}S_{2n-1}(x)$
$\log 2\sin \pi\tau$	$\zeta'(s)$	$\psi(x)$

Remark 1. As in the above scheme, if $a(n)$ is multiplicative, then $F(s)$ has an Euler product, and this new principle of Hecke-like correspondence provides us with a powerful tool for attacking the problem of determining the relation between f and F. A more thorough study in this direction will be conducted elsewhere.

Although we state here the Fourier cosine series only, bearing in mind an application to Farey series, we may consider more general Fourier series in this respect.

We now state the main result on Euler product.

Theorem 2. *Let f be given as the Fourier series* (12)*:*

$$f(u) = \sum_{n=1}^{\infty} c(n) \cos 2\pi n u.$$

Then if $c(n)$ is multiplicative, $\sum_{n=1}^{\infty} |c(n)| < \infty$, and $f(0) = \sum_{n=1}^{\infty} c(n) \neq 0$, then we have for F given by (16)

$$F(s) = G(1)G(s)\widetilde{F}(s),$$

where

$$G(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^{s-1}} = \prod_p G_p(s),$$

with $G_p(s)$ denoting

$$G_p(s) = \sum_{m=0}^{\infty} \frac{c(p^m)}{p^{m(s-1)}},$$

and

$$\widetilde{F}(s) = \prod_p \frac{G_p(s)^{-1} - p^{1-s} G_p(1)^{-1}}{1 - p^{1-s}},$$

where in the product p runs through all primes.

Example 2.

(i) $c(n) = \mu(n) n^{-z}$, $\Re z > 1$.

$$G(s) = \frac{1}{\zeta(s+z-1)}, \quad \widetilde{F}(s) = \prod_p \Big(1 - (p^z - 1)^{-1} (p^{s+z-1} - 1)^{-1}\Big).$$

(ii) $c(n) = \phi(n) n^{-z}$, $\Re z > 2$.

$$G(s) = \frac{\zeta(s+z-2)}{\zeta(s+z-1)}, \quad \widetilde{F}(s) = \prod_p \left(1 + \frac{p-1}{(p^z-1)(p^{s+z-1}-1)}\right).$$

(iii) $c(n) = \sigma_w(n) n^{-z}$, $\Re z > 1$, $\Re w < \Re z - 1$.

$$G(s) = \zeta(s+z-1)\zeta(s+z-w-1), \quad \widetilde{F}(s) = \frac{1}{\zeta(s+2z-w-1)}.$$

Remark 2. In view of the special form of the expression (16), Theorem 2 is a finer decomposition than those obtained in [4] and [5], where the power arguments case and powers of arithmetic functions case are considered, respectively.

3. Gap Fourier series as Weierstrass', Riemann's, and Takagi's function

In this section we consider core functions f given as gap (lacunary) Fourier series, involving two kinds of parameters (α, b) or (κ, l) $((\boldsymbol{\kappa}, \boldsymbol{l}))$, where the first parameter in Greek alphabet can be complex while the last one in alphabet is restricted to positive integers.

The first gap Fourier series that we consider is a Weierstrass function

$$f(u) = f_{\alpha,b}(u) = \sum_{n=0}^{\infty} \alpha^n \cos 2\pi b^n u$$

with $0 < |\alpha| < 1$ and $b \in \mathbb{N}$, $b > 1$. We can treat only the special case $\beta = p^m$, a prime power.

Theorem 3 (Weierstrass function). *Let p be a fixed prime, $m \in \mathbb{N}$, and let $f(u) = f_{\alpha,p^m}(u)$ be a Weierstrass function*

$$f(u) = \sum_{n=0}^{\infty} \alpha^n \cos 2\pi p^{mn} u, \quad 0 < |\alpha| < 1.$$

Then

$$F(s) = \frac{1 + \alpha \sum_{n=1}^{m-1} p^{n(1-s)}}{(1-\alpha)(1 - \alpha p^{m(1-s)})}.$$

Further,

(i) *if $m = 1$ and $0 < |\alpha| < p^{-1/2}$; if $p = 2$, $m \geq 3$ or $p = 3$, $m \geq 4$, and $0 < |\alpha| \leq (p^{1/2} - 1)/(p^{m/2} - p^{1/2})$; or if $p = 2$, $m = 2$ or $p = 3$, $m = 2, 3$ or $p = 5$, $m \geq 2$, and $0 < |\alpha| \leq p^{-m/2}$, then we have*

$$RH \Longleftrightarrow E_f(x) = O\big(x^{\frac{1}{2}+\varepsilon}\big),$$

(ii) *if $m = 1$ and $p^{-1/2} < |\alpha| < 1$; if $p = 2$, $m \geq 3$ or $p = 3$, $m \geq 4$, and $(p^{1/2} - 1)/(p^{m/2} - p^{1/2}) < |\alpha| \leq p^{-m/2}$; or if $p = 2$, $m = 2$ or $p = 3$, $m = 2, 3$ or $p = 5$, $m \geq 2$, and $p^{-m/2} < |\alpha| \leq (p^{1/2} - 1)/(p^{m/2} - p^{1/2})$, then we have*

$$RH \Longleftrightarrow E_f(x) = \frac{x^{1+\frac{\log \alpha}{m \log p}}}{(1-\alpha) m \log p} + O\big(x^{\frac{1}{2}+\varepsilon}\big).$$

Theorem 4. *Let $f_{\kappa,l}(u)$ be a gap Fourier series*

$$f_{\kappa,l}(u) := \sum_{n=1}^{\infty} \frac{1}{n^\kappa} \cos 2\pi n^l u$$

for $\Re\kappa > 1$, $l \in \mathbb{N}$. Then
(i) *$F_{\kappa,l}$ can be decomposed as*

$$F_{\kappa,l}(s) = \zeta(\kappa)\zeta(ls+\kappa-l)\widetilde{F}_{\kappa,l}(s), \tag{18}$$

with $\widetilde{F}_{\kappa,l}$ having an Euler product

$$\widetilde{F}_{\kappa,l}(s) = \prod_p \left(1 + p^{-\kappa} \sum_{n=1}^{l-1} p^{n(1-s)}\right). \tag{19}$$

(ii) *If $2\Re\kappa \geq l+2+\max\{\lambda-1, 0\}$, where*

$$\lambda = 0 \quad \text{or} \quad \frac{2\log\left(1+2^{-\frac{1}{2}}\right)\left(1-2^{-\frac{l-1}{2}}\right)}{\log 2}$$

according as $1 \leq l \leq 3$ or $l \geq 4$, then we have

$$RH \iff E_{f_{\kappa,l}}(x) = O\big(x^{\frac{1}{2}+\varepsilon}\big).$$

(iii) *For $l+1+\max\{\lambda, 0\} \leq 2\Re\kappa < l+2+\max\{\lambda-1, 0\}$, we have*

$$RH \iff E_{f_{\kappa,l}}(x) = \frac{\zeta(\kappa)\widetilde{F}_{\kappa,l}\left(\frac{1}{l}(l-\kappa+1)\right)}{(l-\kappa+1)\zeta\left(\frac{1}{l}(l-\kappa+1)\right)} x^{\frac{1}{l}(l-\kappa+1)} + O\big(x^{\frac{1}{2}+\varepsilon}\big).$$

In particular, for $\kappa = 2$, $l = 3$

$$RH \iff E_{f_{2,3}}(x) = \frac{\zeta(2)\widetilde{F}_{2,3}\left(\frac{2}{3}\right)}{2\zeta\left(\frac{2}{3}\right)} x^{\frac{2}{3}} + O\big(x^{\frac{1}{2}+\varepsilon}\big),$$

where

$$\widetilde{F}_{2,3}\left(\frac{2}{3}\right) = \prod_p \left(1 + p^{-\frac{4}{3}} + p^{-\frac{5}{3}}\right).$$

Corollary. *For $\kappa = l = 2$, $f(u)$ is Riemann's function $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} \cos 2\pi n^2 u$, and*

$$RH \iff E_f(x) = O\big(x^{\frac{1}{2}+\varepsilon}\big).$$

Theorem 5. *For $\boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbb{C}^2$, $\boldsymbol{l} = (l_1, l_2) \in \mathbb{N}^2$, let $f_{\boldsymbol{\kappa},\boldsymbol{l}}(u)$ be the gap-Fourier series:*

$$f(u) = f_{\boldsymbol{\kappa},\boldsymbol{l}}(u) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2^{m\kappa_1}(2n+1)^{\kappa_2}} \cos 2\pi 2^{ml_1}(2n+1)^{l_2} u,$$

with $\Re\kappa_1 > 0$, $\Re\kappa_2 > 1$, l_1, $l_2 \in \mathbb{N}$. Then
(i) *$F_{\boldsymbol{\kappa},\boldsymbol{l}}(s)$ can be decomposed, with F_{κ_2,l_2} as given by* (18)*, as follows:*

$$F_{\boldsymbol{\kappa},\boldsymbol{l}}(s) = \frac{1-2^{-\kappa_2}}{1-2^{-\kappa_1}} \cdot \frac{1-2^{-l_2 s-\kappa_2+l_2}}{1-2^{-l_1 s-\kappa_1+l_1}} \cdot \frac{1+2^{-\kappa_1}\sum_{n=1}^{l_1-1} 2^{n(1-s)}}{1+2^{-\kappa_2}\sum_{n=1}^{l_2-1} 2^{n(1-s)}} F_{\kappa_2,l_2}(s).$$

(ii) *If $2\Re\kappa_2 \geq l_2+2$ and $2\Re\kappa_1 \geq l_1+1+\max\{\lambda_1, 0\}$, with*

$$\lambda_1 = 0 \quad \text{or} \quad \frac{2\log\left(1+2^{-\frac{1}{2}}\right)\left(1-2^{-\frac{l_1-1}{2}}\right)}{\log 2}$$

according as $1 \leq l_1 \leq 3$ or $l_1 \geq 4$, then we have

$$RH \iff E_f(x) = O\big(x^{\frac{1}{2}+\varepsilon}\big).$$

(iii) *For $l_2+1 \leq 2\Re\kappa_2 < l_2+2$ and $2\Re\kappa_1 \geq l_1+1+\max\{\lambda_1, 0\}$, where λ_1 is as in* (i)*, we have*

$$RH \iff E_f(x) = (\textit{Main-term}) + O\big(x^{\frac{1}{2}+\varepsilon}\big),$$

where

$$(\textit{Main-term}) = \frac{1-2^{-\kappa_2}}{1-2^{-\kappa_1}} \cdot \frac{1-2^{-l_2 s_0-\kappa_2+l_2}}{1-2^{-l_1 s_0-\kappa_1+l_1}} \cdot \frac{1+2^{-\kappa_1}\sum_{n=1}^{l_1-1} 2^{n(1-s_0)}}{1+2^{-\kappa_2}\sum_{n=1}^{l_2-1} 2^{n(1-s_0)}}$$
$$\times \frac{\zeta(k_2)\widetilde{F}_{\kappa_2,l_2}(s_0)}{l_2 s_0 \zeta(s_0)} x^{s_0},$$

$\widetilde{F}_{\kappa,l}(s)$ as given by (19)*, and $s_0 = \frac{1}{l_2}(l_2 - \kappa_2 + 1)$.*

Corollary. *For $\kappa_1 = l_1 = l_2 = 1$, $\kappa_2 = 2$, $f(u)$ is Takagi's function* $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{2^m(2n+1)^2}\cos 2\pi 2^m(2n+1)u$, *and*

$$RH \iff E_f(x) = O\big(x^{\frac{1}{2}+\varepsilon}\big).$$

4. Proofs

Lemma 1 (Generalization of Lemma 2 [15])**.** *Suppose that*

$$(*) \qquad E_f(x) = cx^{\beta} + O\big(x^{\frac{1}{2}+\varepsilon}\big) \quad \text{for every } \varepsilon > 0$$

holds. Then

(i) *functions $F(s)$ and $F(s)/\zeta(s)$ are regular for $\sigma > \frac{1}{2}$, $s \neq \beta, 1$,*

(ii) *in the half-plane $\sigma > \frac{1}{2}$, $\zeta(s)$ can have zeros only at possible zeros of $F(s)$.*

Lemma 2. (i) ([16, Lemma 1]) *Let $0 < \alpha \leq 1$, $0 \leq \xi_1 < \xi_2 \leq 1$ and $f \in \Lambda_\alpha$. Then*

$$\sum_{\xi_1 < \rho_\nu \leq \xi_2} f(\rho_\nu) = \Phi(x)\int_{\xi_1}^{\xi_2} f(t)dt + O\left(x^{2-\alpha}\right).$$

On the RH, the error term can be reduced to $O\big(x^{2-\frac{3}{2}\alpha+\varepsilon}\big)$ for every $\varepsilon > 0$, where the range of integration is to be replaced by $[\eta_1, \eta_2]$, $\eta_i = h(\xi_i)/\Phi(x)$ $(i = 1, 2)$, for $\alpha > \frac{2}{3}$. And in particular, for any α, $0 < \alpha \leq 1$, we have on the RH

$$\sum_{\xi_1 < \rho_\nu \leq \xi_2} f(\rho_\nu) = \Phi(x)\int_{\eta_1}^{\eta_2} f(t)dt + O\big(x^{2-\frac{3}{2}\alpha+\varepsilon}\big).$$

(ii) ([17, Lemma 5]) *Let $\{a(n)\}$ be a complex sequence and $F(s)$ denote the generating function of $a(n)$. Suppose that $F(s)$ and $a(n)$ satisfy the following conditions:*

(a) *$F(s)$ is absolutely convergent for $\sigma > \sigma_a$ with $\sigma_a \leq 1$,*

(b) *$F(s)$ is continued to an analytic function to the half-plane $\sigma > \alpha$ with finitely many singularities with $\frac{1}{2} \leq \alpha < 1$,*

(c) *$F(s) \ll |t|^{\kappa+\varepsilon}$ for some $\kappa \geq 0$ and every $\varepsilon > 0$ uniformly in the region $\alpha < \sigma \leq 1$, $|t| \geq t_0 > 0$,*

(d) $(\mu * a)(n) \ll n^{\beta+\varepsilon}$ *for sum* β, $0 \le \beta\ (\le \sigma_a)$,

(e) *there exists a non-negative number* θ *satisfying*

$$\sum_{n=1}^{\infty} \frac{|\mu * a|(n)}{n^\sigma} \ll (\sigma - 1)^{-\theta} \quad \text{as } \sigma \to 1.$$

Then, on the RH(α), *we have*

$$\sum_{n \le x} (\mu * a)(n) = \frac{1}{2\pi i} \int_C \frac{F(s)}{s\zeta(s)} x^s ds + O\left(x^{\omega+\varepsilon}\right),$$

where

$$\omega = \min_{0 \le \xi \le 1} \{\max\{\beta + 1 - \xi,\ 1 + (\kappa - 1)\xi,\ \alpha + \kappa\xi\}\},$$

and the contour C *encircle all singularities of* $F(s)/\zeta(s)$ *in the strip* $\alpha < \sigma < 1$.

In particular, in the special cases of $\kappa = 0$ *and* $\beta = 0$ *we have* $\omega = \max\{\alpha,\ \beta\}$, *and* $\omega = \alpha + \kappa(1 - \alpha)$, *respectively.*

Proof of Theorem 2. We make use of the second equality of (16). Writing

$$F(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^{s-1}} \sigma_{s-1}(n) = \prod_p \sum_{l=0}^{\infty} \frac{c(p^l)}{(p^l)^{s-1}} \sigma_{s-1}(p^l),$$

we apply the well-known formula

$$\sigma_{s-1}(p^l) = \frac{1 - (p^{l+1})^{s-1}}{1 - p^{s-1}}.$$

Then

$$\begin{aligned} F(s) &= \prod_p \frac{1}{1 - p^{s-1}} \left(G_p(s) - G_p(1) p^{s-1}\right) \\ &= \prod_p G_p(s) G_p(1) \frac{G_p(s)^{-1} - p^{1-s} G_p(1)^{-1}}{1 - p^{1-s}}, \end{aligned}$$

whence the result follows. □

Proof of Theorem 3. The proof consists of two stages, i.e. proving that f is of Lipschitz class and obtaining the Euler product, whence establishing the zero free region of $F(s)$.

First we shall prove that
Case (1) if $0 < \alpha < \frac{1}{p}$, then $f \in \Lambda_1$,
Case (2) if $\alpha = p^{-\tau_0}$, $0 < \tau_0 < 1$, then $f \in \Lambda_{\tau_0}$
and Case (3) if $\alpha = \frac{1}{p}$, then $f \in \Lambda_{1-\varepsilon}$, for every $\varepsilon > 0$.

For $x, y \in [0,1]$, we may classify the difference $|x-y|$ according to powers of p (the extremal case $(x,y) = (0,1)$, $(1,0)$ being clear):

$$\frac{1}{p^{m+1}} \le |x-y| < \frac{1}{p^m}, \quad m = 0, 1, 2, \dots$$

Let $0 < \tau \le 1$ be fixed and let n be the variable of summation for $f(u)$. Then for $0 \le n \le m$, we have

$$|\cos 2\pi p^n u - \cos 2\pi p^m v| \le 2\pi p^n |u-v| < 2\pi p^{n-(1-\tau)m} |u-v|^\tau,$$

while for $n > m$,

$$|\cos 2\pi p^n u - \cos 2\pi p^m v| \le 2 \le 2p^{m+1}|u-v| \le 2p^{m\tau+1}|u-v|^\tau.$$

Hence, on taking the sum of resulting geometric progression, we infer that

$$|f(u)-f(v)| < 2\left(\frac{\pi}{1-\lambda p} p^{-(1-\tau)m} - \frac{\pi\lambda p}{1-\lambda p}(\lambda p^\tau)^m + \frac{\lambda p}{1-\lambda}(\lambda p^\tau)^m\right)|u-v|^\tau \quad \text{if } \lambda p \ne 1,$$

and

$$|f(u)-f(v)| < 2\left((m+1)\pi + \frac{p}{p-1}\right) p^{-(1-\tau)m}|u-v| \quad \text{if } \lambda p = 1.$$

Then, in Case (1)

$$|f(u)-f(v)| \le M|u-v|^\tau,$$

with

$$M = 2\left(\frac{\pi}{1-\lambda p} + \frac{\lambda p}{1-\lambda}\right).$$

Since we can take $\tau = 1$, we conclude that $f \in \Lambda_1$.

In Case (2),

$$|f(u) - f(v)| \le M|u - v|^{\tau_0}$$

with

$$M = 2\left(\frac{\pi\alpha p}{\alpha p - 1} + \frac{\alpha p}{1-\alpha}\right),$$

so that $f \in \Lambda_{\tau_0}$.

Finally, in Case (3) considering the upper bound of the function

$$g(y) = \left(\pi y + \pi + \frac{p}{p-1}\right) p^{-(1-\tau)y} \quad (y \ge 0),$$

we obtain

$$|f(u) - f(v)| \le M_\varepsilon |u - v|^{1-\varepsilon}$$

for every $\varepsilon > 0$, where

$$M_\varepsilon = \max\left\{\pi + \frac{p}{p-1}, \frac{\pi}{\varepsilon e \log p} p^{(1+\frac{2}{\pi})}\right\}.$$

Hence we get necessarily part of our assertion.

Euler product representation (18) follows immediately from Theorem 2 on noting that with the prescribed p

$$c(n) = \begin{cases} 0 & \text{if } n \ne p^{ml} \\ \alpha^l & \text{if } n = p^{ml}, \end{cases}$$

so that

$$G(s) = G_p(s) = \frac{1}{1 - \alpha p^{m(1-s)}}.$$

It remains to examine a possible simple pole and non-vanishingness of $F(s)$ for $\sigma > \frac{1}{2}$.

The case where $F(s)$ has a simple pole at $s = 1 + \frac{\log \alpha}{m \log p}$ is stated in (ii) while the case where $F(s)$ is regular (for $\sigma > \frac{1}{2}$) is stated in (i).

The condition is stated in terms of inequalities in α at the end of each if-clause, and non vanishingness conditions are stated preceding them. □

Proof of Theorem 4. Since

$$c(n) = \begin{cases} n^{-\kappa/l} & \text{if } n = m^l \\ 0 & \text{otherwise}, \end{cases}$$

we have

$$G_p(s) = \sum_{m=0}^{\infty} p^{(l-ls-\kappa)m} = \frac{1}{1-p^{l-ls-\kappa}},$$

so that

$$\widetilde{F}(s) = \widetilde{F}_{\kappa,l}(s) = \prod_p \left(1 + p^{-\kappa} \frac{p^{1-s} - p^{l(1-s)}}{1-p^{1-s}}\right)$$

as asserted in (19).

The proof is similar in nature to that of Theorem 3, i.e. we distinguish the cases where $F(s)$ has a simple pole at $s = \frac{l-\kappa+1}{l}$ or not and find the conditions for nonvanishing of $F(s)$, save for the necessity part to which we apply an abelian theorem.

We shall illustrate the proof by the case $\kappa = 2$ and $l = 3$.

In this case we have the Euler product decomposition

$$F(s) = F_{2,3}(s) = \zeta(2)\zeta(3s-1)\widetilde{F}_{2,3}(s), \tag{20}$$

where

$$\widetilde{F}_{2,3}(s) = \prod_p \left(1 + p^{-s-1} + p^{-2s}\right),$$

which is absolutely convergent for $\sigma > \frac{1}{2}$ and has no zeros there.

On the RH we have $F(s) \ll |t|^{\varepsilon}$ for $\sigma > \frac{1}{2}$, $|t| > t_0 > 0$, $\varepsilon > 0$ arbitrary. Hence by Lemma 2, (ii) (noting that $F(s)$ has a simple pole at $s = \frac{2}{3}$)

$$E_f(x) = \frac{\zeta(2)}{2\zeta(\frac{2}{3})} \widetilde{F}_{2,3}\left(\frac{2}{3}\right) x^{\frac{2}{3}} + O\left(x^{\omega+\varepsilon}\right),$$

where $\omega = \max\{\gamma, \frac{1}{2}\}$ with γ such that

$$(a * \mu)(n) \ll n^{\gamma+\varepsilon}.$$

By (20) we obtain

$$(a * \mu)(n) \ll \sigma_{1/3}(n) \ll n^{\frac{1}{3}+\varepsilon},$$

so that we may take $\gamma = \frac{1}{3}$ and the necessity part follows.

The sufficiency part follows from Lemma 1 since $F(s) \neq 0$ for $\sigma > \frac{1}{2}$.
□

PROOF of Corollary to Theorem 4. This follows in the same way as that for Theorem 4 on account of

$$F(s) = F_{2,2}(s) = \frac{\zeta(2)\zeta(2s)\zeta(s+1)}{\zeta(2s+2)} \neq 0$$

for $\sigma > \frac{1}{2}$. □

PROOF of Theorem 5 and its Corollary. The Euler product decomposition follows from

$$c(n) = \begin{cases} 2^{-\kappa_1 n_2} \prod_{p \neq 2} p^{-\kappa_2 n_p} & \text{if } n = 2^{l_1 n_2} \prod_{p \neq 2} p^{l_2 n_p} \\ 0 & \text{otherwise,} \end{cases}$$

and the proof is similar to that for Theorem 4 ($F_{\boldsymbol{\kappa},\boldsymbol{l}}(s) = F_{\kappa,l}(s)$ if $\kappa_1 = \kappa_2 = \kappa$, $l_1 = l_2 = l$).

We shall sketch the proof of Corollary to Theorem 5. In this case

$$F(s) = F_{(1,1),(1,2)}(s) = \frac{3}{2}\frac{1-2^{-s-1}}{1-2^{-s}}\zeta(2)\zeta(s+1),$$

which is non-zero for $\sigma > \frac{1}{2}$.

Hence Lemma 2, (ii) applies. □

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