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On optimal linear congruences for $L_2(k, \chi\omega^{1-k})$

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Dedicated to Professor Kálmán Győry
on the occasion of his 60th birthday

Abstract. Our purpose in the paper is to investigate divisibility properties of 2-adic L-functions attached to quadratic characters at integers. Following UEHARA's ideas we extend the linear congruence relations proved in [6], [8] and [10] (see also [3], [4], [5], [6] and [7]). For any two-element subset L of the set $\{-1, 0, 1, 2\}$ we determine the so-called optimal linear congruence relations for $L_2(k, \chi\omega^{1-k})$, with $k \in L$.

1. Notation

For prime p as usual we denote by $\mathbb{C}_p$ the completion of the algebraic closure of $\mathbb{Q}_p$. $\mathbb{Q}_p$ denotes the field of p-adic numbers. For $a, b \in \mathbb{C}_p$ and $\alpha \in \mathbb{Q}$ the notation $a \equiv b \pmod{p^\alpha}$ means that $|a-b|_p \leq p^{-\alpha}$. $|\cdot|_p$ denotes the normalized (such that $|p|_p = 1/p$) absolute value on $\mathbb{C}_p$. For $a, b \in \mathbb{Z}$ and $\alpha \in \mathbb{N}$ these congruences are the usual congruences for integral rational numbers. We say that $a \in \mathbb{C}_p$ is p-integral if $a \equiv 0 \pmod{p^0}$. For $a \in \mathbb{Q}$, if a is p-integral in the above sense then its denominator is not divisible by p. We say that p-integral number a is divisible by p^α ($\alpha \geq 1$) if $a \equiv 0 \pmod{p^\alpha}$. We write $p^\alpha \mid a$. If for p-integral number a we have

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$a \not\equiv 0 \pmod{p^\alpha}$, we write $p^\alpha \nmid a$ and say that a is not divisible by p^α. For $\alpha \in \mathbb{N}$ if $p^\alpha \mid a$ and $p^{\alpha+1} \nmid a$, we set $p^\alpha \| a$. For $\alpha, \beta \in \mathbb{N}$ and $p^\alpha \| a$, we write $\gcd(p^\beta, a) = p^\alpha$ (resp. p^β) if $\alpha \le \beta$ (resp. $\beta < \alpha$). If $a \equiv b \pmod{p^\beta}$, we have

$$\gcd\left(p^\beta, a\right) = \gcd\left(p^\beta, b\right). \tag{1.1}$$

Moreover if $m, n \in \mathbb{C}_p$ are p-integers not divisible by p, we observe that

$$\gcd\left(p^\beta, a\right) = \gcd\left(p^\beta, \frac{a}{m}\right) = \gcd(p^\beta, an). \tag{1.2}$$

We say that a $(\in \mathbb{C}_2)$ is even if a is 2-integral and divisible by 2. We say that a is odd if a is 2-integral and is not even.

As usual let $\log = \log_p$, $\omega = \omega_p$ denote the p-adic logarithm and the Teichmüller character at p respectively. For a Dirichlet character χ let $L_p(s, \chi)$ be the Kubota–Leopoldt L-function. For details see [9].

For $k \in \mathbb{Z}$ let $l_k = l_{k,p}$ denote the so-called multilogarithms, which are locally analytic functions on the set $\mathbb{C}_p - \{1\}$ defined inductively by $l_0(s) = -s/(1-s)$, $dl_k(s) = l_{k-1}(s)ds/s$ and $\lim_{s\to 0} l_k(s) = 0$. For details, see [1]. Moreover if $k \le 0$, we have $l_k(s) = s(-1)^k R_{-k}(s)/(1-s)^{1-k}$, where $R_n \in \mathbb{Z}[x]$ $(n \ge 0)$ are the so-called Frobenius polynomials defined in [2]. If $k = -1$ we have $l_{-1}(s) = s/(1-s)^2$ in particular. If $k = 1$, we have $l_1(s) = -\log_p(1-s)$.

The main interest of the multilogarithms is that they give the Coleman formulas

$$L_p(k, \chi\omega^{1-k}) = (1 - \chi(p)p^{-k}) \frac{\tau\left(\chi, \zeta_M\right)}{M} \sum_{a=1}^{M-1} \overline{\chi}(a) l_{k,p}\left(\zeta_M^{-a}\right).$$

Here χ is a primitive non-trivial Dirichlet character modulo M and throughout the paper we denote by ζ_M a primitive Mth root of unity in $\mathbb{C}_p$.

For a fundamental discriminant d $(\neq 1)$ as usual we denote by χ_d the associated quadratic character (Kronecker symbol). We set $\chi_1 = 1$. Denote by $\mathcal{T}_d$ the set of all fundamental discriminants dividing d. Throughout the paper, for $t, c \in \mathbb{Z}$ $(t \neq 0,\ c \ge 1)$ we denote by $\nu(t)$ the number of distinct prime factors of t and adopt the notation $\sum_{a=1}^{c}{}'$ to a sum taken over integers a prime to c. As usual ϕ denotes Euler's phi function.

The proofs of the main theorems of the paper (Theorems 1 and 2) are based on the following lemma.

Lemma 1 (see [8, Lemma 1], cf. [6, Lemma 3]). *Let χ be a Dirichlet character modulo $M>1$ and let N be a multiple of M such that $N/M>0$ is a rational square-free integer relatively prime to M. For arbitrary natural number T satisfying $M|T|N$ we assume that $\zeta_T=\zeta_M\zeta_{T/M}$ and set*

$$\mathcal{S}_{k,\chi}(T)=\sum_{a=1}^{T}{}'\chi(a)l_k(\zeta_T^a).$$

Then for any integer k we have

$$\mathcal{S}_{k,\chi}(N)=(-1)^{\nu(N/M)}\prod_{\substack{p|(N/M)\\ p\ \text{prime}}}\bigl(1-\overline{\chi}(p)p^{1-k}\bigr)\mathcal{S}_{k,\chi}(M).$$

2. Quadratic fields

If d is the discriminant of a quadratic field, we denote by $h(d)$, $k_2(d)$, ε_d, resp. $R_2(d)$ the class number, the order of the K_2-group of the integers, the fundamental unit, resp. the second Borel regulator of the field $\mathbb{Q}(\sqrt{d})$. For $k\in\{-1,0,1,2\}$ we have

$$L(k,\chi_d)=\begin{cases}-12w_2^{-1}(d)k_2(d), & \text{if } k=-1 \text{ and } d>1,\\ 2w^{-1}(d)h(d), & \text{if } k=0 \text{ and } d<0,\\ 2d^{-1/2}h(d)\log\varepsilon_d, & \text{if } k=1 \text{ and } d>1,\\ 2R_2(d)|d|^{-3/2}k_2(d), & \text{if } k=2 \text{ and } d<0,\end{cases}$$

where $w(-3)=6$, $w(-4)=4$, $w(d)=2$ if $d<-4$ and $w_2(8)=48$, $w_2(5)=120$, $w_2(d)=24$ if $d>8$. Here $L(s,\chi)$ is the classical, complex Dirichlet L-function attached to χ. In the case when $k=2$ we assume that the so-called Lichtenbaum conjecture for imaginary quadratic fields holds.

Usually, the complex and p-adic formulas differ by an Euler factor. Namely we have

$$L_p(k,\chi_d\omega^{1-k})$$
$$=\begin{cases} -12w_2^{-1}(d)\bigl(1-\chi_d(p)p\bigr)k_2(d), & \text{if } k=-1 \text{ and } d>1,\\ 2w^{-1}(d)\bigl(1-\chi_d(p)\bigr)h(d), & \text{if } k=0 \text{ and } d<0,\\ 2d^{-1/2}\bigl(1-\chi_d(p)p^{-1}\bigr)h(d)_p\log\varepsilon_d, & \text{if } k=1 \text{ and } d>1,\\ 2R_{2,p}(d)|d|^{-3/2}\bigl(1-\chi_d(p)p^{-2}\bigr)k_2(d), & \text{if } k=2 \text{ and } d<0,\end{cases}$$

where by analogy $R_{2,p}(d)$ denotes the second p-adic regulator of the corresponding field $\mathbb{Q}(\sqrt{d})$. In the case when $k=2$ the above equation is the statement of a p-adic analogue of the Lichtenbaum conjecture for imaginary quadratic fields.

3. The numbers $W_{k,e}(n)$

Let k, $n\in\mathbb{Z}$ and $e\in\mathcal{T}_8$. For $n\ge 0$ write

$$\gamma_{n,e}=\begin{cases} -1, & \text{if } n\equiv 1,2 \pmod 4 \text{ and } e\in\mathcal{T}_8-\mathcal{T}_4,\\ 1, & \text{otherwise}\end{cases}$$

and

$$W_{k,e}(n)=\sum_{l=0}^{n}(-1)^{l(k+1)}(2l+1)^{1-k}\gamma_{l,e}\binom{2n+1}{n-l}.$$

The numbers $W_{k,e}(n)$ are 2-integral rational numbers. We have $\operatorname{ord}_2(W_{k,e}(n))\ge n$. For details see [10].

4. Uehara's functions

From now on we assume that $p=2$, $\omega=\omega_2$ and $l_k=l_{k,2}$. For any Dirichlet character ψ modulo f and $k\in\mathbb{Z}$ let $\mathcal{L}_{k,\psi}$ denote the so-called Uehara functions. These functions are defined by

$$\mathcal{L}_{k,\psi}(s)=\frac{1}{2}(-1)^{k+1}\bigl(l_k(s)-l_k(-s)\bigr)\quad (s\ne\pm 1),$$

if ψ is the trivial character, and

$$\mathcal{L}_{k,\psi}(s)=(-1)^{k+1}\frac{\tau(\overline{\psi},\zeta_f)}{f}\sum_{a=1}^{f}{}'\psi(a)l_k(\zeta_f^a s)\quad (s\neq\zeta_f^a)$$

otherwise. For details see [8]. For $\psi=\chi_e$ set $\mathcal{L}_{k,\psi}=\mathcal{L}_{k,e}$.

The proof of the main result of the paper (Theorem 1) is based on the following properties of Uehara's functions implied by the identity of Lemma 1 and proved in [8] and [10].

Lemma 2 (see [6], [8, Lemma 2] and [10, Lemma 1]). *Given any odd integer M, let χ by a primitive Dirichlet character modulo M. Suppose that N is an odd multiple of M such that N/M (>0) is a rational square-free integer relatively prime to M. Let ψ be a primitive Dirichlet character being either trivial or of even conductor coprime to N. Assume that for arbitrary natural number T satisfying $M|T|N$ we have $\zeta_T=\zeta_M\zeta_{T/M}$. Then for any integer k we have*

$$\begin{aligned}&\frac{\tau(\overline{\chi},\zeta_M)}{M}\sum_{a=1}^{N}{}'\chi(a)\mathcal{L}_{k,\psi}(\zeta_N^a)\\ &=(-1)^{\nu(N/M)}\prod_{\substack{p|(N/M)\\ p\ \text{prime}}}\big(1-\overline{\chi\psi}(p)p^{1-k}\big)L_2\big(k,\overline{\chi\psi}\omega^{1-k}\big),\end{aligned}$$

unless $k=1$ and the characters χ and ψ are trivial, in which case we have

$$\sum_{a=1}^{N}{}'\mathcal{L}_{k,\psi}(\zeta_N^a)=\begin{cases}-(\log_2 N)/2, & \textit{if } N \textit{ is a prime number},\\ 0, & \textit{otherwise}.\end{cases}$$

Remark. In the formulation of Lemma 2 of [8] there is a small error, which implies the same error in Lemma 1 of [10]. The right hand sides of the identities of the lemmas should be multiplied by $(-1)^{k+1}$.

Lemma 3. *Let c (>1) be an odd natural number. If $k\neq 0$, 1 we have*

$$\sum_{a=1}^{c}{}'l_k(\zeta_c^a)=(-1)^{k+1+\nu(c)}\big(1-2^{-k}\big)^{-1}\prod_{\substack{p|c\\ p\ \text{prime}}}\big(1-p^{1-k}\big)L_2\big(k,\omega^{1-k}\big).$$

If $k=0$ or 1 we have

$$\sum_{a=1}^{c}{}' l_k(\zeta_c^a) = \begin{cases} -\dfrac{1}{2}\phi(c), & \text{if } k=0,\\ -\log_2 c, & \text{if } k=1 \text{ and } c \text{ is a prime number},\\ 0, & \text{otherwise} \end{cases}$$

Proof. Given $r \in \mathbb{N}$ we have

$$\frac{1}{r}\sum_{\zeta^r=1} l_k(\zeta z) = \frac{l_k(z^r)}{r^k}$$

(see [1, Proposition 6.1]). Applying this formula with $r=2$ we obtain

$$\mathcal{L}_{k,1}(s) = (-1)^{k+1}\big(l_k(s) - 2^{-k} l_k(s^2)\big) \quad (s \neq \pm 1).$$

Hence we have

$$\sum_{a=1}^{c}{}' l_k(\zeta_c^a) = (-1)^{k+1}\big(1-2^{-k}\big)^{-1}\sum_{a=1}^{c}{}'\mathcal{L}_{k,1}(\zeta_c^a)$$

because

$$\begin{aligned}\big(1-2^{-k}\big)\sum_{a=1}^{c}{}' l_k(\zeta_c^a) &= (-1)^{k+1}\sum_{a=1}^{c}{}'(-1)^{k+1}\big(l_k(\zeta_c^a) - l_k(\zeta_c^{2a})\big)\\ &= (-1)^{k+1}\sum_{a=1}^{c}{}'\mathcal{L}_{k,1}(\zeta_c^a).\end{aligned}$$

Thus Lemma 3 in the case when $k \neq 0$ follows easily from Lemma 2.

If $k=0$ we have

$$\sum_{a=1}^{c}{}' l_0(\zeta_c^a) = \sum_{a=1}^{c}{}'\frac{\zeta_c^a}{1-\zeta_c^a} = \sum_{a=1}^{c}{}'\frac{1}{1-\zeta_c^a} - \phi(c) = \frac{1}{2}\phi(c) - \phi(c) = -\frac{1}{2}\phi(c),$$

which completes the proof. □

Lemma 4 (cf. [6, Lemma 2]). *Given d $(\neq 1)$ an odd fundamental discriminant we have*

$$\sum_{a=1}^{|d|}\chi_d(a) l_0(\zeta_{|d|}^a) = \begin{cases} -\dfrac{|d|h(d)}{\tau\big(\chi_d,\zeta_{|d|}\big)}, & \text{if } d<0,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. By the definition of l_0 we have

$$\sum_{a=1}^{|d|} \chi_d(a) l_0(\zeta_{|d|}^a) = \sum_{a=1}^{|d|} \frac{\chi_d(a)\zeta_{|d|}^a}{1-\zeta_{|d|}^a} = \sum_{a=1}^{|d|} \frac{\chi_d(a)}{1-\zeta_{|d|}^a} - \sum_{a=1}^{|d|} \chi_d(a).$$

Hence and from Lemma 2 [6] the identity of the hypothesis of Lemma 4 follows immediately. □

In Lemmas 5 and 6 ξ ($\neq 1$) denotes a primitive Nth root of unity, where N is an odd natural number.

Lemma 5 (see [6] and [8, Lemma 4]). *For any $e \in \mathcal{T}_8$ write $\alpha = \operatorname{sgn} e$ and set*

$$w_\alpha = \frac{\alpha\xi}{1+\alpha\xi^2}.$$

Then we have

$$\mathcal{L}_{-1,e}(\xi) = \sum_{k=0}^{\infty} (4\alpha)^k w_\alpha^{2k+1}, \quad \mathcal{L}_{0,e}(\xi) = \omega_{-\alpha},$$

$$\mathcal{L}_{1,e}(\xi) = \sum_{k=0}^{\infty} \frac{(4\alpha)^k \omega_\alpha^{2k+1}}{2k+1}, \quad \mathcal{L}_{2,e}(\xi) = \sum_{k=0}^{\infty} \frac{(-16\alpha)^k \omega_{-\alpha}^{2k+1}}{(2k+1)^2} \binom{2k}{k}^{-1},$$

if $e \in \mathcal{T}_4$, and

$$\mathcal{L}_{-1,e}(\xi) = -\sum_{k=0}^{\infty} (2\alpha)^k (2k-1) \omega_\alpha^{2k+1}, \qquad \mathcal{L}_{0,e}(\xi) = \sum_{k=0}^{\infty} (-2\alpha)^k \omega_{-\alpha}^{2k+1},$$

$$\mathcal{L}_{1,e}(\xi) = \sum_{k=0}^{\infty} \frac{(2\alpha)^k \omega_\alpha^{2k+1}}{2k+1},$$

$$\mathcal{L}_{2,e}(\xi) = \sum_{k=0}^{\infty} \frac{(-16\alpha)^k \omega_{-\alpha}^{2k+1}}{(2k+1)^2} \binom{2k}{k}^{-1} \sum_{l=0}^{k} \binom{2l}{l} 2^{-3l},$$

if $e \in \mathcal{T}_8 - \mathcal{T}_4$.

Remark. Uehara in a letter to the author has observed that the formulas for $\mathcal{L}_{-1,e}(\xi)$ and $\mathcal{L}_{2,e}(\xi)$ given in the above lemma can be deduced easily from his formulas for $\mathcal{L}_{0,e}(\xi)$, $\mathcal{L}_{1,e}(\xi)$, and differential properties of Coleman's multilogarithms. The details of the proof are left to the reader as an exercise.

Lemma 6 (see [10, Lemma 3]). *For any* $e \in \mathcal{T}_8$ *and* $m \in \mathbb{Z}$ *write* $\alpha = (-1)^{m+1} \operatorname{sgn} e$ *and let*

$$w_\alpha = \frac{\alpha\xi}{1+\alpha\xi^2}.$$

Then we have

$$\mathcal{L}_{m,e}(\xi) = \sum_{k=0}^{\infty} \frac{\alpha^k W_{m,e}(k)}{2k+1} w_\alpha^{2k+1}.$$

5. Some special sequences

Let K be a finite non-empty subset of the rational integers. We will consider linear combinations of Uehara's functions at ξ with 2-adic integral coefficients

$$x = \{x_{k,e}\}_{(k,e)\in K\times\mathcal{T}_8} \subseteq \mathbb{C}_2.$$

For any $L \subseteq K$ the x is said to be defined on L if $x_{k,e} = 0$ for $k \notin L$. Let

$$\alpha_k = \binom{2k}{k}^{-1} \quad \text{and} \quad \beta_k = \binom{2k}{k}^{-1} \sum_{l=0}^{k} \binom{2l}{l} 2^{-3l}.$$

Given 2-adic integers $a_{k,e}(n)$ ($\in \mathbb{C}_2$) with $k \in K$, $e \in \mathcal{T}_8$, $n \geq 0$ we consider some sequences of linear combinations of $x_{k,e}$ of the form

$$y_n(x) = \sum_{(k,e)\in K\times\mathcal{T}_8} a_{k,e}(n) x_{k,e}, \quad n \geq 0. \tag{5.3}$$

For any $L \subseteq K$ the sequence $(y_n)_{n\geq 0}$ of this form is said to be defined on L, if the sum is taken over $(k,e) \in L \times \mathcal{T}_8$.

For $x = \{x_{k,e}\}_{(k,e)\in K\times\mathcal{T}_8}$ we consider two sequences $z = (z_n)_{n\geq 0}$ and $u = (u_n)_{n\geq 0}$ of the form (5.3). The sequences are defined on $K = \{-1, 0, 1, 2\}$ in the former case and on any finite subset K of $\mathbb{Z}$ in the latter case by

$$z_0 = \sum_{(k,e)\in K\times\mathcal{T}_8} x_{k,e}, \qquad z_1 = 2 \sum_{\substack{(k,e)\in K\times\mathcal{T}_8 \\ \operatorname{sgn} e = (-1)^k}} x_{k,e},$$

$$\begin{aligned} z_{2l+\varrho} = 2^{l+\varrho}\Big(& 2^l(2l+1)^2\big((1-\varrho)x_{-1,1} + x_{-1,-4}\big) \\ & - (2l-1)(2l+1)^2\big((1-\varrho)x_{-1,8} + x_{-1,-8}\big) \\ & + (2l+1)^2\big((1-\varrho)x_{0,-8} + x_{0,8}\big) \\ & + 2^l(2l+1)\big((1-\varrho)x_{1,1} + x_{1,-4}\big) \\ & + (2l+1)\big((1-\varrho)x_{1,8} + x_{1,-8}\big) \\ & + 2^{3l}\alpha_l\big((1-\varrho)x_{2,-4} + x_{2,1}\big) \\ & + 2^{3l}\beta_l\big((1-\varrho)x_{2,-8} + x_{2,8}\big)\Big), \end{aligned}$$

if $l \geq 1$, $\varrho \in \{0,1\}$, and

$$u_{2l+\varrho} = 2^{\varrho}\sum_{k,e}(-1)^{l(k+1)}(2l+1)^{1-k}\gamma_{l,e}x_{k,e}, \quad l \geq 0, \ \varrho \in \{0,1\},$$

where the sum in the latter case is taken over all $(k,e) \in K \times \mathcal{T}_8$ if $\varrho = 0$, and over $(k,e) \in K \times \mathcal{T}_8$ with $\operatorname{sgn} e = (-1)^k$ if $\varrho = 1$.

Let $y = (y_n)_{n\geq 0}$ be a sequence of the form (5.3). Let $c = c(y)$ be a non-negative number such that there exist 2-adic integers $x_{k,e}$ not all even satisfying

$$y_n(x) \equiv 0 \pmod{2^c}, \quad n \geq 0,$$

and if for some 2-adic integers $x_{k,e}$ we have

$$y_n(x) \equiv 0 \pmod{2^{c+1}}, \quad n \geq 0,$$

then all the numbers $x_{k,e}$ are even.

Lemma 7 (see [8, Lemma 5]). *Let $K = \{-1,0,1,2\}$ and let L be a non-empty subset of K. Write $c(L) = c(z)$, where $z = (z_n)_{n\geq 0}$ is the sequence given above, defined on L. Then we have*

$$c(L) = 12, \ 9, \ 5, \ \textit{resp. } 2,$$

if $\operatorname{card}(L) = 4, 3, 2,$ *resp.* 1, *unless* $L = \{-1,1\}$ *or* $\{0,2\}$, *in which cases*

$$c(L) = 6.$$

Lemma 8 (see [10, Lemma 5]). *Let $m \geq 1$ be an integer and let*

$$K = \{-m+2, -m+3, \ldots, 1\}.$$

Then we have

$$c(u_n) = 3m - 1 + \operatorname{ord}_2\big((m-1)!\big).$$

Remark. Lemma 8 is also valid for any set consisting of m consecutive integers. In order to prove it we apply the same reasoning as in the proof of Lemma 5 [10].

6. Linear combinations of $\mathcal{L}_{k,e}(\xi)$

Recall that N is an odd natural number and ξ ($\neq 1$) is a primitive Nth root of unity in $\mathbb{C}_2$. Given 2-adic integers $\{x_{k,e}\}_{(k,e)\in K\times\mathcal{T}_8} \subseteq \mathbb{C}_2$ not all even, defined on a non-empty subset L of K, our purpose is to evaluate the linear combinations

$$\sum_{(k,e)\in K\times\mathcal{T}_8} x_{k,e}\mathcal{L}_{k,e}(\xi),$$

modulo powers of 2. In order to obtain the congruences stated in Lemma 9 we appeal to Lemmas 5 and 7. Combining the obtained congruences with Lemmas 1 and 2 we shall derive some new congruences for linear combinations of the values of 2-adic L-functions $L_2(k, \chi\omega^{1-k})$ with arbitrary 2-adic integral coefficients, where χ are primitive quadratic Dirichlet characters.

Lemma 9 (see [8, Lemma 5]). *Set $K = \{-1, 0, 1, 2\}$. Let $x_{k,e}$ ($k \in K$, $e \in \mathcal{T}_8$) be 2-adic integers not all even defined on a non-empty subset L of K. Then we have*

$$\sum_{(k,e)\in L\times\mathcal{T}_8} x_{k,e}\mathcal{L}_{k,e}(\xi) \equiv 0 \pmod{2^\lambda},$$

where 2^λ is the greatest common divisor of

$$2^{c(L)} \text{ and } z_n, \quad 0 \leq n \leq \max\big(2c(L)-4, 2\big),$$

and

$$c(L) = 12,\ 9,\ 5,\ \textit{resp. } 2,$$

if $\operatorname{card}(L) = 4, 3, 2$, *resp.* 1, *unless* $L = \{-1, 1\}$ *or* $\{0, 2\}$, *in which cases*

$$c(L) = 6.$$

PROOF. We first observe that for n even

$$2z_n = z_{n+1} + \tilde{z}_{n+1},$$

where the $\tilde{z}_{n+1}$ comes from z_{n+1} by replacing $x_{k,-4}$ (resp. $x_{k,1}$, $x_{k,-8}$ or $x_{k,8}$) by $x_{k,1}$ (resp. $x_{k,-4}$, $x_{k,8}$ or $x_{k,-8}$).

In [8, Lemma 5] the congruence of Lemma 9 was proved modulo the greatest common divisor of $2^{c(L)}$ and z_n, $0 \leq n \leq 2c(L)-2$. Now it suffices to use the congruences

$$z_{2l+1} \equiv 2^{l+1}\eta \pmod{2^{l+2}}, \quad \tilde{z}_{2l+1} \equiv 2^{l+1}\tilde{\eta} \pmod{2^{l+2}},$$
$$z_{2l} \equiv 2^{l}\left(\eta + \tilde{\eta}\right) \pmod{2^{l+1}},$$

where $l \geq 1$ and

$$\eta = x_{-1,-8} + x_{0,8} + x_{1,-8} + x_{2,8}.$$

These congruences follow immediately by the definition of the $z_{2l+\varrho}$. Indeed we have

$$\begin{aligned} z_{2l+\varrho} \equiv 2^{l+\varrho}\Big(&\big((1-\varrho)x_{-1,8} + x_{-1,-8}\big) + \big((1-\varrho)x_{0,-8} + x_{0,8}\big) \\ &+ \big((1-\varrho)x_{1,8} + x_{1,-8}\big) + \big((1-\varrho)x_{2,-8} + x_{2,8}\big)\Big) \pmod{2^{l+\varrho+1}} \end{aligned}$$

because $\operatorname{ord}_2\left(2^{3l}\alpha_l\right) \geq 2l$ and $\operatorname{ord}_2\left(2^{3l}\beta_l\right) = 0$.

By the above, we have

$$z_{2c(L)-2} \equiv 2^{c(L)-1}\left(\eta + \tilde{\eta}\right) \pmod{2^{c(L)}}$$
$$z_{2c(L)-3} \equiv 2^{c(L)-1}\eta \pmod{2^{c(L)}}$$
$$z_{2c(L)-4} \equiv 2^{c(L)-2}\left(\eta + \tilde{\eta}\right) \pmod{2^{c(L)-1}},$$
$$z_{2c(L)-5} \equiv 2^{c(L)-2}\eta \pmod{2^{c(L)-1}},$$

provided $c(L) > 2$. Therefore we may ignore $z_{2c(L)-2}$ and $z_{2c(L)-3}$ if $c(L) > 2$. □

Appealing to Lemmas 6 and 8 we obtain:

Lemma 10 (see [10, Lemma 6]). *Let $m \geq 1$ be an integer and let*

$$K = \{-m+2, -m+3, \dots, 1\}.$$

Let $x_{k,e}$ $(k, \in K, e \in \mathcal{T}_8)$ be integers in $\mathbb{C}_2$ not all even. Then we have

(i)
$$\sum_{(k,e)\in K\times\mathcal{T}_8} x_{k,e}\mathcal{L}_{k,e}(\xi) \equiv 0 \pmod{2^\lambda},$$

where 2^λ is the greatest common divisor of

$$2^{c(u_n)} \text{ and } u_n, \qquad 0 \leq n \leq 4m-1,$$

(ii) *for an arbitrary integer s*

$$\sum_{(k,e)\in K\times\mathcal{T}_8} x_{k,e}\mathcal{L}_{k+s,e}(\xi) \equiv 0 \pmod{2^\lambda}.$$

7. Main theorems

In this section we extend linear congruence relations proved in [8] and [10]. We follow Uehara's ideas from [6] and give a further generalization of the Gras–Uehara type congruence for linear combinations of the values of 2-adic L-functions $L_2(k, \chi\omega^{1-k})$, where χ is a quadratic Dirichlet character. We restrict our attention to the cases when k is taken over an arbitrary non-empty subset L of the set $K = \{-1, 0, 1, 2\}$ or when k is taken over an arbitrary finite set of consecutive integers. These cases were considered in [8] and [10] respectively. It appears to be still an open problem to find the Gras–Uehara type congruence when k is taken over any finite subset of the rational integers.

Let d be an odd fundamental discriminant and let $m > 1$ be a natural number. Throughout the paper let Ψ, $\Theta : \mathbb{N} \to \mathbb{C}_2$ be multiplicative functions such that $\Psi(s) \equiv \Theta(s) \equiv 1 \pmod 2$ if $s \mid m$. Let $\delta_{X,Y}$ denote the Kronecker delta function, that is, $\delta_{X,Y} = 1$ if $X = Y$ and is zero otherwise. For $k \in \mathbb{Z}$ and $e \in \mathcal{T}_8$ we write

$$L_2^{[m,\Theta]}(k, \chi_{ed}\omega^{1-k}) = 0$$

if $d = e = k = 1$, and

$$L_2^{[m,\Theta]}(k, \chi_{ed}\omega^{1-k}) = \Bigg(\prod_{\substack{p \mid m \\ p \text{ prime}}} \big(1 - \chi_{ed}(p)\Theta(p)p^{1-k}\big) - \delta_{d,1} \prod_{\substack{p \mid m \\ p \text{ prime}}} \big(1 - \Theta(p)\big)\Bigg) L_2(k, \chi_{ed}\omega^{1-k})$$

otherwise. Set

$$L_{2,*}^{[m,\Theta]}(k, \chi_d\omega^{1-k}) = \begin{cases} h(d), & \text{if } k = 0 \text{ and } d < 0, \\ 0, & \text{if } k = 0 \text{ and } d > 0, \\ \big(1 - \chi_d(2)2^{-k}\big)^{-1} L_2^{[m,\Theta]}(k, \chi_d\omega^{1-k}), & \text{otherwise.} \end{cases}$$

If $\Theta(s) = 1$ for $s \mid m$, we have $L_2^{[m,\Theta]}(k, \chi_{ed}\omega^{1-k}) = L_2^{[m]}(k, \chi_{ed}\omega^{1-k})$ and

$$L_2^{[m]}(k, \chi_{ed}\omega^{1-k}) = \begin{cases} 0, & \text{if } d = e = k = 1, \\ \displaystyle\prod_{\substack{p \mid m \\ p \text{ prime}}} \big(1 - \chi_{ed}(p)p^{1-k}\big) L_2(k, \chi_e\omega^{1-k}), & \text{otherwise.} \end{cases}$$

Now we are ready to extend the main theorems of the papers [8] and [10]. Let $m, s > 1$ be square-free natural numbers with $s \mid m$. We shall apply the following identity

$$\sum_{t \mid s} \Theta(t) \prod_{\substack{p \mid (s/t) \\ p \text{ prime}}} (1 - \Theta(p)) \prod_{\substack{p \mid t \\ p \text{ prime}}} (1 - \Phi(p)) = \prod_{\substack{p \mid s \\ p \text{ prime}}} (1 - \Phi(p)\Theta(p)), \tag{7.4}$$

see [6, (3.1)].

Theorem 1 (cf. [8, Main Theorem], [10, Theorem])**.** *Let $m > 1$ be a square-free odd natural number having ν prime factors and let Ψ, $\Theta : \mathbb{N} \to \mathbb{C}_2$ be multiplicative functions satisfying $\Psi(s) \equiv \Theta(s) \equiv 1 \pmod 2$ if $s \mid m$. Let K have the same meaning as in Lemma 9 (resp. Lemma 10)* *and let* $x=\{x_{k,e}\}_{(k,e)\in K\times\mathcal{T}_8}$ *be a set of 2-adic integers not all even. Set*

$$\Lambda_1(m,\Theta)=-\frac{1}{2}\sum_{\substack{p\mid m\\ p\ \text{prime}}}\Theta(p)\log_2 p\prod_{\substack{q\mid(m/p)\\ q\ \text{prime}}}(1-\Theta(q)).$$

Then the number

$$\Lambda(x,m,\Psi,\Theta):=\sum_{(k,e)\in K\times\mathcal{T}_8}x_{k,e}\sum_{d\in\mathcal{T}_m}\Psi(|d|)L_2^{[m,\Theta]}\big(k,\chi_{ed}\omega^{1-k}\big)+x_{1,1}\Lambda_1(m,\Theta)$$

is a 2-adic integer divisible by $2^{\nu+\lambda}$*, where* λ *has the same meaning as in Lemma 9 if* $K=\{-1,0,1,2\}$ *and* x *is defined on a non-empty finite subset* L *of* K *(resp. Lemma 10 if* K *is a finite set of consecutive integers).*

Proof. Write

$$\Lambda_2(x,m,\Theta)=\prod_{\substack{p\mid m\\ p\ \text{prime}}}\big(1-\Theta(p)\big)\sum_{\substack{(k,e)\in K\times\mathcal{T}_8\\ (k,e)\neq(1,1)}}x_{k,e}L_2\big(k,\chi_e\omega^{1-k}\big).$$

and

$$L_2'\big(k,\chi_{ed}\omega^{1-k}\big)=\begin{cases}0, & \text{if } e=d=k=1,\\ L_2\big(k,\chi_{ed}\omega^{1-k}\big), & \text{otherwise.}\end{cases}$$

We proceed in the same manner as in the proof of the Main Theorem in [8] (resp. the Theorem in [10]). Making use of (7.4), for any multiplicative function $\Phi:\mathbb{N}\to\mathbb{C}_2$ and fixed u, s with $u\mid s$ we obtain

$$\begin{aligned}\Theta^{-1}(u)\sum_{u|t|s}\Theta(t)\prod_{\substack{p\mid(s/t)\\ p\ \text{prime}}}(1-\Theta(p))\prod_{\substack{p\mid(t/u)\\ p\ \text{prime}}}(1-\Phi(p))\\ =\prod_{\substack{p\mid(s/u)\\ p\ \text{prime}}}(1-\Phi(p)\Theta(p)).\end{aligned}\tag{7.5}$$

This follows from (7.4) by a simple induction on the number of prime factors of s/u. We observe that for any functions f and g

$$\sum_{d\mid m}f(d)\sum_{c\mid d}g(c)h(d,c)=\sum_{d\mid m}g(d)\sum_{d|c|m}f(c)h(c,d).\tag{7.6}$$

Therefore we have

$$\begin{aligned}
&\Lambda(x, m, \Psi, \Theta) - x_{1,1}\Lambda_1(m, \Theta) + \Lambda_2(x, m, \Theta)\\
&\quad = \sum_{(k,e)\in K\times\mathcal{T}_8} x_{k,e} \sum_{d\in\mathcal{T}_m} \Psi(|d|) \prod_{\substack{p|(m/d)\\ p \text{ prime}}} \big(1 - \Theta(p)\chi_{ed}(p)p^{1-k}\big) L_2'\big(k, \chi_{ed}\omega^{1-k}\big)\\
&\quad = \sum_{(k,e)\in K\times\mathcal{T}_8} x_{k,e} \sum_{d\in\mathcal{T}_m} \Psi(|d|)\Theta^{-1}(|d|) L_2'\big(k, \chi_{ed}\omega^{1-k}\big)\\
&\qquad \times \sum_{\substack{c\in\mathcal{T}_m\\ pd\in\mathcal{T}_c}} \Theta(|c|) \prod_{\substack{p|(m/c)\\ p \text{ prime}}} (1 - \Theta(p)) \prod_{\substack{p|(c/d)\\ p \text{ prime}}} (1 - \chi_{ed}(p)p^{1-k})\\
&\quad = \sum_{(k,e)\in K\times\mathcal{T}_8} x_{k,e} \sum_{d\in\mathcal{T}_m} \Theta(|d|) \prod_{\substack{p|(m/d)\\ p \text{ prime}}} (1 - \Theta(p))\\
&\qquad \times \sum_{c\in\mathcal{T}_d} \Psi(|c|)\Theta^{-1}(|c|) \prod_{\substack{p|(d/c)\\ p \text{ prime}}} \big(1 - \chi_{ec}(p)p^{1-k}\big) L_2'\big(k, \chi_{ec}\omega^{1-k}\big).
\end{aligned}$$

Consequently appealing to Lemma 2 we obtain

$$\begin{aligned}
&\Lambda(x, m, \Psi, \Theta)\\
&\quad = \sum_{1\neq d\in\mathcal{T}_m} \Theta(|d|)\mu(|d|) \prod_{\substack{p|(m/d)\\ p \text{ prime}}} (1 - \Theta(p)) \sum_{a=1}^{|d|}{}' \Big(\sum_{\substack{k\in K\\ e\in\mathcal{T}_8}} x_{k,e}\mathcal{L}_{k,e}\big(\zeta_{|d|}^a\big)\Big)\\
&\qquad \times \Big(\sum_{c\in\mathcal{T}_d} \mu(|c|)\Psi(|c|)\Theta^{-1}(|c|)\tau\big(\chi_c, \zeta_{|c|}\big)|c|^{-1}\chi_c(a)\Big)\\
&\quad = \sum_{1\neq d\in\mathcal{T}_m} \Theta(|d|)\mu(|d|) \prod_{\substack{p|(m/d)\\ p \text{ prime}}} (1 - \Theta(p)) \sum_{a=1}^{|d|}{}' \Big(\sum_{\substack{k\in K\\ e\in\mathcal{T}_8}} x_{k,e}\mathcal{L}_{k,e}\big(\zeta_{|d|}^a\big)\Big)\\
&\qquad \times \Big(\prod_{\substack{p|d\\ p \text{ prime}}} \big(1 - \tau(\chi_{p^*}, \zeta_p)p^{-1}\Psi(p)\Theta^{-1}(p)\chi_{p^*}(a)\big)\Big),
\end{aligned}$$

where $p^* = (-1)^{(p-1)/2}p$ and $\zeta_{|d|} = \prod_{\substack{p|d\\ p \text{ prime}}} \zeta_p$.

Now Theorem 1 follows from Lemma 9 when $K = \{-1, 0, 1, 2\}$ or from Lemma 10 when K is a set of consecutive integers. □

The Main Theorem in [8] and Theorem in [10] are special cases of Theorem 1 when $\Theta(s) = 1$ for $s \mid m$.

We now extend Theorem 2 [6] (a supplement of Theorem 1 [6]). Let m (> 1) be a square-free odd natural number. Denote by $I(m)$ the set of $k \in \mathbb{Z}$ such that $l_k(\zeta_c^a)$ are 2-adic integers for any c and a with $c \mid m$, $c \neq 1$, $1 \leq a \leq c$ and $\gcd(a, c) = 1$. By definition, we have $1 \in I(m)$ and $r \in I(m)$ for any integer $r \leq 0$. The question whether $I(m) = \mathbb{Z}$ remains to be open.

Theorem 2 (cf. [6, Theorem 2]). *Let $m > 1$ be a square-free odd natural number having ν prime factors and let $\Psi, \Theta : \mathbb{N} \to \mathbb{C}_2$ be multiplicative functions satisfying $\Psi(s) \equiv \Theta(s) \equiv 1 \pmod 2$ if $s \mid m$. Set*

$$\Lambda_{0,*}(m, \Theta) = \frac{1}{2}\Big(\prod_{\substack{p \mid m \\ p \text{ prime}}} \big(1 - \Theta(p)p\big) - \prod_{\substack{p \mid m \\ p \text{ prime}}} \big(1 - \Theta(p)\big)\Big)$$

and

$$\Lambda_{1,*}(m, \Theta) = \sum_{\substack{p \mid m \\ p \ \ prime}} \Theta(p) \log_2 p \prod_{\substack{q \mid (m/p) \\ q \text{ prime}}} (1 - \Theta(q)).$$

For $k \in I(m)$ the number

$$\Lambda_*(k, m, \Psi, \Theta) := \sum_{d \in \mathcal{T}_m} \Psi(|d|) L_{2,*}^{[m,\Theta]}\big(k, \chi_d \omega^{1-k}\big) + \delta_{k,0} \Lambda_{0,*}(m, \Theta) + \delta_{k,1} \Lambda_{1,*}(m, \Theta)$$

is a 2-adic integer divisible by 2^ν.

Proof. Write

$$\Lambda'(k, m, \Theta) = \begin{cases} \big(1 - 2^{-k}\big)^{-1} L_2^{[m,\Theta]}\big(k, \omega^{1-k}\big), & \text{if } k \neq 0, 1, \\ \Lambda_{k,*}(m, \Theta), & \text{otherwise} \end{cases}$$

and

$$\Lambda''(k, m, \Psi, \Theta) = \sum_{d \in \mathcal{T}_m} \Psi(|d|) \prod_{\substack{p \mid (m/d) \\ p \text{ prime}}} \big(1 - \Theta(p) \chi_d(p) p^{1-k}\big) L_2''\big(k, \chi_d \omega^{1-k}\big),$$

where

$$L_2''\big(k,\chi_d\omega^{1-k}\big)=\begin{cases} h(d), & \text{if } k=0 \text{ and } d<0,\\ 0, & \text{if } k=0 \text{ and } d>0,\\ & \text{or } k\neq 0 \text{ and } d=1,\\ \big(1-\chi_d(2)2^{-k}\big)^{-1}L_2\big(k,\chi_d\omega^{1-k}\big), & \text{otherwise.}\end{cases}$$

We first observe that

$$\Lambda_*(k,m,\Psi,\Theta)=\Lambda'(k,m,\Theta)+\Lambda''(k,m,\Psi,\Theta).$$

On the other hand, by virtue of (7.4) we have

$$\Lambda'(k,m,\Theta)=\big(1-2^{-k}\big)^{-1}\sum_{\substack{d\in\mathcal{T}_m\\ d\neq 1}}\Theta(|d|)\prod_{\substack{p|(m/d)\\ p\ \text{prime}}}\big(1-\Theta(p)\big)$$
$$\times\prod_{\substack{p|d\\ p\ \text{prime}}}\big(1-p^{1-k}\big)L_2\big(k,\omega^{1-k}\big),$$

if $k\neq 0,1$ and

$$\Lambda'(0,m,\Theta)=\frac{1}{2}\sum_{\substack{d\in\mathcal{T}_m\\ d\neq 1}}(-1)^{\nu(d)}\Theta(|d|)\phi(|d|)\prod_{\substack{p|(m/d)\\ p\ \text{prime}}}\big(1-\Theta(p)\big).$$

Moreover by virtue of (7.5) we have

$$\Lambda''(k,m,\Psi,\Theta)=\sum_{d\in\mathcal{T}_m}\Psi(|d|)L_2''\big(k,\chi_d\omega^{1-k}\big)\Theta^{-1}(|d|)\sum_{d|c|m}\Theta(|c|)$$
$$\times\prod_{\substack{p|(m/c)\\ p\ \text{prime}}}\big(1-\Theta(p)\big)\prod_{\substack{p|(c/d)\\ p\ \text{prime}}}\big(1-\chi_d(p)p^{1-k}\big),$$

and so in view of (7.6) we obtain

$$\Lambda''(k,m,\Psi,\Theta)=\sum_{d\in\mathcal{T}_m}\Theta(|d|)\prod_{\substack{p|(m/d)\\ p\ \text{prime}}}\big(1-\Theta(p)\big)$$
$$\times\sum_{c\in\mathcal{T}_d}\Psi(|c|)\Theta^{-1}(|c|)\prod_{\substack{p|(d/c)\\ p\ \mathit{prime}}}\big(1-\chi_c(p)p^{1-k}\big)L_2''\big(k,\chi_c\omega^{1-k}\big).$$

Therefore appealing to Lemmas 3 and 4 we deduce that

$$\Lambda'(k,m,\Theta)=(-1)^{k+1}\sum_{\substack{d\in\mathcal{T}_m\\ d\neq 1}}(-1)^{\nu(d)}\Theta(|d|)\prod_{\substack{p|(m/d)\\ p\ \text{prime}}}\bigl(1-\Theta(p)\bigr)\sum_{b=1}^{|d|}{}'\,l_k\bigl(\zeta_{|d|}^b\bigr)$$

and

$$\Lambda''(k,m,\Psi,\Theta)=(-1)^{k+1}\sum_{\substack{d\in\mathcal{T}_m\\ d\neq 1}}\Theta(|d|)\prod_{\substack{p|(m/d)\\ p\ \text{prime}}}\bigl(1-\Theta(p)\bigr)$$

$$\times\sum_{c\in\mathcal{T}_d}\Psi(|c|)\Theta^{-1}(|c|)\frac{\tau\bigl(\chi_c,\zeta_{|c|}\bigr)}{|c|}\prod_{\substack{p|(d/c)\\ p\ \text{prime}}}\bigl(1-\chi_c(p)p^{1-k}\bigr)\sum_{b=1}^{|c|}{}'\,\chi_c(b)l_k\bigl(\zeta_{|c|}^b\bigr).$$

Thus in view of Lemma 1 we have

$$\begin{aligned}\Lambda_*(k,m,\Psi,\Theta)&=(-1)^{k+1}\sum_{\substack{d\in\mathcal{T}_m\\ d\neq 1}}(-1)^{\nu(d)}\Theta(|d|)\prod_{\substack{p|(m/d)\\ p\ \text{prime}}}\bigl(1-\Theta(p)\bigr)\\
&\qquad\times\sum_{c\in\mathcal{T}_d}\Psi(|c|)\Theta^{-1}(|c|)\mu(|c|)\frac{\tau\bigl(\chi_c,\zeta_{|c|}\bigr)}{|c|}\sum_{b=1}^{|d|}{}'\,\chi_c(b)l_k\bigl(\zeta_{|d|}^b\bigr)\\
&=(-1)^{k+1}\sum_{\substack{d\in\mathcal{T}_m\\ d\neq 1}}(-1)^{\nu(d)}\Theta(|d|)\prod_{\substack{p|(m/d)\\ p\ \text{prime}}}\bigl(1-\Theta(p)\bigr)\sum_{b=1}^{|d|}{}'\,l_k\bigl(\zeta_{|d|}^b\bigr)\\
&\qquad\times\sum_{c\in\mathcal{T}_d}\mu(|c|)\Psi(|c|)\Theta^{-1}(|c|)\frac{\tau\bigl(\chi_c,\zeta_{|c|}\bigr)}{|c|}\chi_c(b)\\
&=(-1)^{k+1}\sum_{\substack{d\in\mathcal{T}_m\\ d\neq 1}}\Theta(|d|)\prod_{\substack{p|(m/d)\\ p\ \text{prime}}}\bigl(1-\Theta(p)\bigr)\sum_{b=1}^{|d|}{}'\,l_k\bigl(\zeta_{|d|}^b\bigr)\\
&\qquad\times\prod_{\substack{p|d\\ p\ \text{prime}}}\Bigl(\tau\bigl(\chi_{p^*},\zeta_p\bigr)p^{-1}\Psi\Theta^{-1}(p)\chi_{p^*}(b)-1\Bigr),\end{aligned}$$

which proves Theorem 2. □

8. Optimal linear congruences

The congruences in the hypothesis of Theorem 1

$$\sum_{(k,e)\in K\times\mathcal{T}_8} x_{k,e} \sum_{d\in\mathcal{T}_m} \Psi(|d|) L_2^{[m,\Theta]}\big(k, \chi_{ed}\omega^{1-k}\big)$$

$$+ x_{1,1}\Lambda_1(m,\Theta) \equiv 0 \pmod{2^{\nu+\lambda}}$$

are said to be optimal if $\lambda = c(L)$ (resp. $\lambda = c(u_n)$). The 2-adic integers $x_{k,e}$ ($k \in K$, $e \in \mathcal{T}_8$) determining an optimal linear congruence are called optimal for K. For example, the congruences proved in [4], [7] or resp. [5] are optimal for $K = \{0\}$, $K = \{-1, 0\}$ or resp. $K = \{-m, \dots, -1, 0\}$ ($m \geq 0$).

Optimal linear congruences exist for any non-empty subset L of $K = \{-1, 0, 1, 2\}$ and when K is a finite subset of consecutive integers. Such a congruence was given explicitly in the proof of Lemma 5 in [8] in the former case and inductively in the proof of Lemma 6 in [10] in the latter case.

9. Applications of Theorem 1

When $L = \{0, 1\}$ Theorem 1 gives the congruences of Gras [3] and Uehara [6] for class numbers of quadratic fields which are modulo $2^{\nu+\lambda}$, where $\lambda \leq 5$. When $L = \{-1, 0\}$ (resp. $L = \{0\}$) we obtain congruences for the same objects as those in [7] (resp. [4]). The obtained congruences are modulo $2^{\nu+\lambda}$, where $\lambda \leq 6$ (resp. $\lambda \leq 2$). When $2 \in L$ the congruences implied by Theorem 1 are quite new and especially interesting. They produce, via a 2-adic version of the Lichtenbaum conjecture, some new congruences for the conjectured orders of K_2-groups of the integers of imaginary quadratic fields. We present these congruences in a general form in Theorem 3.

For the discriminant $\mathcal{D}$ of a quadratic field, we write

$$H(\mathcal{D}) = L_2\big(k, \chi_{\mathcal{D}}\omega^{1-k}\big) \quad \big(\text{resp. } K_2(\mathcal{D}) = 2L_2\big(k, \chi_{\mathcal{D}}\omega^{1-k}\big)\big),$$

if $k = 0$, $\mathcal{D} < 0$ or $k = 1$, $\mathcal{D} > 1$ (resp. $k = -1$, $\mathcal{D} > 1$ or $k = 2$, $\mathcal{D} < 0$). We have

$$H(\mathcal{D}) = \begin{cases} 2w^{-1}(\mathcal{D})\big(1 - \chi_{\mathcal{D}}(2)\big)h(\mathcal{D}), & \text{if } \mathcal{D} < 0, \\ \big(2 - \chi_{\mathcal{D}}(2)\big)\mathcal{D}^{-1/2}h(\mathcal{D})\log_2\varepsilon_{\mathcal{D}}, & \text{if } \mathcal{D} > 1, \end{cases}$$

and

$$K_2(\mathcal{D}) = \begin{cases} -24w_2^{-1}(\mathcal{D})\bigl(1-\chi_{\mathcal{D}}(2)2\bigr)k_2(\mathcal{D}), & \text{if } \mathcal{D}>1, \\ \bigl(4-\chi_{\mathcal{D}}(2)\bigr)|\mathcal{D}|^{-3/2}R_{2,2}(\mathcal{D})k_2(\mathcal{D}), & \text{if } \mathcal{D}<0. \end{cases}$$

In the formula for $K_2(\mathcal{D})$ when $\mathcal{D} < 0$ we assume that the 2-adic Lichtenbaum conjecture for imaginary quadratic fields holds. Now we are ready to extend results of [8, Applications]. We rewrite Theorem 1 with $K = \{-1, 0, 1, 2\}$ in the form:

Theorem 3 (cf. [8, Applications]). *Let $m > 1$ be a square-free odd natural number having ν prime factors and let $\Theta, \Psi : \mathbb{N} \to \mathbb{C}_2$ be multiplicative functions such that $\Theta(s) \equiv \Psi(s) \equiv 1 \pmod 2$ if $s \mid m$. Set $K = \{-1, 0, 1, 2\}$ and let L be a non-empty subset of K. Given a set $x = \{x_{k,e}\}_{(k,e)\in K\times\mathcal{T}_8}$ of 2-adic integers not all even defined on L, set*

$$\Lambda = \Lambda_{-1} + \Lambda_0 + \Lambda_1 + \Lambda_2 + \Lambda'_{-1} + \Lambda'_1,$$

where

$$\Lambda_{-1} = \frac{1}{2}\sum_{e\in\mathcal{T}_8} x_{-1,e} \sum_{\substack{d\in\mathcal{T}_m \\ ed>1}} \Psi(|d|) \times \Bigl(\prod_{\substack{p\mid m \\ p \text{ prime}}} \bigl(1-\chi_{ed}(p)\Theta(p)p^2\bigr) - \delta_{d,1} \prod_{\substack{p\mid m \\ p \text{ prime}}} \bigl(1-\Theta(p)\bigr)\Bigr) K_2(ed),$$

$$\Lambda_0 = \sum_{e\in\mathcal{T}_8} x_{0,e} \sum_{\substack{d\in\mathcal{T}_m \\ ed<0}} \Psi(|d|) \times \Bigl(\prod_{\substack{p\mid m \\ p \text{ prime}}} \bigl(1-\chi_{ed}(p)\Theta(p)p\bigr) - \delta_{d,1} \prod_{\substack{p\mid m \\ p \text{ prime}}} \bigl(1-\Theta(p)\bigr)\Bigr) H(ed),$$

$$\Lambda_1 = \sum_{e\in\mathcal{T}_8} x_{1,e} \sum_{\substack{d\in\mathcal{T}_m \\ ed>1}} \Psi(|d|) \times \Bigl(\prod_{\substack{p\mid m \\ p \text{ prime}}} \bigl(1-\chi_{ed}(p)\Theta(p)\bigr) - \delta_{d,1} \prod_{\substack{p\mid m \\ p \text{ prime}}} \bigl(1-\Theta(p)\bigr)\Bigr) H(ed),$$

$$\Lambda_2 = \frac{1}{2}\sum_{e\in\mathcal{T}_8} x_{2,e} \sum_{\substack{d\in\mathcal{T}_m \\ ed<0}} \Psi(|d|)$$

$$\times\Big(\prod_{\substack{p\mid m\\ p\text{ prime}}}\big(1-\chi_{ed}(p)\Theta(p)p^{-1}\big)-\delta_{d,1}\prod_{\substack{p\mid m\\ p\text{ prime}}}\big(1-\Theta(p)\big)\Big)K_2(ed),$$

$$\Lambda'_{-1}=\frac{1}{12}x_{-1,1}\Big(\prod_{\substack{p\mid m\\ p\text{ prime}}}\big(1-\Theta(p)p^2\big)-\prod_{\substack{p\mid m\\ p\text{ prime}}}\big(1-\Theta(p)\big)\Big),$$

$$\Lambda'_1=-\frac{1}{2}x_{1,1}\sum_{\substack{p\mid m\\ p\text{ prime}}}\Theta(p)\log_2 p\prod_{\substack{q\mid(m/p)\\ q\text{ prime}}}\big(1-\Theta(q)\big).$$

Then the number Λ is a 2-adic integer divisible by $2^{\nu+\lambda}$, where λ has the same meaning as in Theorem 1.

10. The case $L = \{0,1\}$

Hardy and Williams [4] discovered a new type of linear congruence relating class numbers of imaginary quadratic fields. A general linear congruence relating class numbers and units both of real and imaginary quadratic fields was discovered by Gras [3]. Gras derived his congruence using 2-adic measure theory. Uehara [6] reproved Gras' congruence using elementary 2-adic arguments. Both Gras and Uehara used the 2-adic analogue of Dirichlet's class number formulas. Urbanowicz and Wójcik [8] and Wójcik [10] indicated how Uehara's techniques may be used to obtain more general congruences among the values of 2-adic L-functions. Gras and Uehara's congruences are special cases of Theorems 1 and 2.

Theorem 4 (see [6, Theorem 1]). *Let $m>1$ be an odd square-free integer having ν prime factors, and let $\Theta,\Psi:\mathbb{N}\to\mathbb{C}_2$ be multiplicative functions such that $\Psi(s)\equiv\Theta(s)\equiv 1 \pmod 2$ for any divisor $s\mid m$. In the notation of Theorem 3, for any 2-adic integers $x_{0,e}$, $x_{1,e}$ ($e\in\mathcal{T}_8$) not all even we have*

$$\sum_{e\in\mathcal{T}_8}x_{0,e}\sum_{\substack{d\in\mathcal{T}_m\\ ed<0}}\Psi(|d|)\Big(\prod_{\substack{p\mid m\\ p\text{ prime}}}\big(1-\chi_{ed}(p)\Theta(p)p\big)-\delta_{d,1}\prod_{\substack{p\mid m\\ p\text{ prime}}}\big(1-\Theta(p)\big)\Big)H(ed)$$
$$+\sum_{e\in\mathcal{T}_8}x_{1,e}\sum_{\substack{d\in\mathcal{T}_m\\ ed>1}}\Psi(|d|)\Big(\prod_{\substack{p\mid m\\ p\text{ prime}}}\big(1-\chi_{ed}(p)\Theta(p)\big)-\delta_{d,1}\prod_{\substack{p\mid m\\ p\text{ prime}}}\big(1-\Theta(p)\big)\Big)H(ed)$$
$$-\frac{1}{2}x_{1,1}\sum_{\substack{p\mid m\\ p\text{ prime}}}\Theta(p)\log_2 p\prod_{\substack{q\mid(m/p)\\ q\text{ prime}}}\big(1-\Theta(q)\big)\equiv 0 \pmod{2^{\nu+\lambda}},$$

where 2^λ is the greatest common divisor of the eight integers s_i $(0 \le i \le 7)$ defined by

$$\begin{aligned}
s_0 &= x_{0,-8} + x_{0,-4} + x_{0,1} + x_{0,8} + x_{1,-8} + x_{1,-4} + x_{1,1} + x_{1,8},\\
s_1 &= 2\bigl(x_{0,1} + x_{0,8} + x_{1,-8} + x_{1,-4}\bigr),\\
s_2 &= 2\bigl(3x_{0,-8} + 3x_{0,8} + x_{1,-8} + 2x_{1,-4} + 2x_{1,1} + x_{1,8}\bigr),\\
s_3 &= 4\bigl(3x_{0,8} + x_{1,-8} + 2x_{1,-4}\bigr),\\
s_4 &= 4\bigl(5x_{0,-8} + 5x_{0,8} + x_{1,-8} + 4x_{1,-4} + 4x_{1,1} + x_{1,8}\bigr),\\
s_5 &= 8\bigl(x_{0,8} + x_{1,-8}\bigr),\\
s_6 &= 8\bigl(x_{0,-8} + x_{0,8} - x_{1,-8} - x_{1,8}\bigr),\\
s_7 &= 32.
\end{aligned}$$

Remark. The proof of Theorem 4 is straightforward. We see at once that $\gcd(z_i, 32) = \gcd(s_i, 32)$, $0 \le i \le 6$, which is clear from (1.1) and (1.2) (with $p = 2$).

Theorem 4 is the main result of [6]. This theorem and its supplement stated in [6, Theorem 2] include the congruences proved in [3, Théorèmes (1.3), (1.4)] and [4]. For details and other applications see [6].

In fact Uehara has provided a general method of producing such congruences. It is a simple matter to determine linear congruence relations with given λ. We will look more closely at the case when $\lambda = 5$.

Corollary 1. *The congruence in the hypothesis of Theorem 4 is optimal if and only if*

$$\begin{aligned}
x_{0,-8} &= a,\\
x_{0,-4} &= a + 32b - 16c - 24d + 4e + 4f + 2g,
\end{aligned}$$

$$
\begin{aligned}
x_{0,1} &= -a + 16c + 16d - 4e - 4f - 2g + 2h,\\
x_{0,8} &= -a + 16d - 4f + 2h,\\
x_{1,-8} &= a - 16d + 4f + 4g - 2h,\\
x_{1,-4} &= a - 16d + 4e + 4f - 2g - 2h,\\
x_{1,1} &= -a - 8d - 4e + 4f + 2g,\\
x_{1,8} &= -a + 32d - 8f - 4g,
\end{aligned}
$$

where $a, b, c, d, e, f, g, h \in \mathbb{C}_2$ are integers with a odd.

Proof. The congruence in the hypothesis of Theorem 4 is valid modulo $2^{\nu+5}$ if and only if

$$
\begin{gathered}
s_0 = 32b,\ s_1 = 32c,\ s_2 = 32d,\ s_3 = 32e,\\
s_4 = 32f,\ s_5 = 32g,\ s_6 = 32h
\end{gathered}
\tag{10.7}
$$

for some integers $b, c, d, e, f, g, h \in \mathbb{C}_2$. Taking $x_{0,-8} = a$ we obtain a system of seven linear equations with seven unknowns $x_{0,-4}$, $x_{0,1}$, $x_{0,8}$, $x_{1,-8}$, $x_{1,-4}$, $x_{1,1}$, $x_{1,8}$ and determinant -8. An easy computation gives the formulas of Corollary 1 at once. □

Corollary 2. *If the congruence in the hypothesis of Theorem 4 is optimal then all the $x_{0,e}$, $x_{1,e}$ $(e \in \mathcal{T}_8)$ are odd. None of these coefficients can vanish in particular.*

11. The case $L = \{-1, 0\}$

In this case the obtained congruences extend those of [7] for the orders of K_2-groups of the integers of real quadratic fields and class numbers of imaginary quadratic fields. We leave it to the reader to show that Theorem 5 implies the Theorem in [7]. In the case when $L = \{-1, 0\}$ we have $c(L) = 5$ and the congruences are valid modulo $2^{\nu+\lambda+1}$, where $\lambda \le 5$.

Theorem 5. *Let $m > 1$ be an odd square-free integer having ν prime factors, and let $\Theta, \Psi : \mathbb{N} \to \mathbb{C}_2$ be multiplicative functions such that $\Psi(s) \equiv$*

$\Theta(s) \equiv 1 \pmod 2$ *for any divisor* $s \mid m$. *In the notation of Theorem 3, for any 2-adic integers* $x_{-1,e}$, $x_{0,e}$ $(e \in \mathcal{T}_8)$ *not all even we have*

$$\sum_{e\in\mathcal{T}_8} x_{-1,e} \sum_{\substack{d\in\mathcal{T}_m\\ ed>1}} \Psi(|d|)\Big(\prod_{\substack{p|m\\ p \text{ prime}}}\big(1-\chi_{ed}(p)\Theta(p)p^2\big)-\delta_{d,1}\prod_{\substack{p|m\\ p \text{ prime}}}\big(1-\Theta(p)\big)\Big)K_2(ed)$$
$$+2\sum_{e\in\mathcal{T}_8} x_{0,e} \sum_{\substack{d\in\mathcal{T}_m\\ ed<0}} \Psi(|d|)\Big(\prod_{\substack{p|m\\ p \text{ prime}}}\big(1-\chi_{ed}(p)\Theta(p)p\big)-\delta_{d,1}\prod_{\substack{p|m\\ p \text{ prime}}}\big(1-\Theta(p)\big)\Big)H(ed)$$
$$+\frac{1}{6}x_{-1,1}\Big(\prod_{\substack{p|m\\ p \text{ prime}}}\big(1-\Theta(p)p^2\big)-\prod_{\substack{p|m\\ p \text{ prime}}}\big(1-\Theta(p)\big)\Big) \equiv 0 \pmod{2^{\nu+\lambda+1}},$$

where 2^λ *is the greatest common divisor of the eight integers* s_i $(0 \le i \le 7)$ *defined by*

$$s_0 = x_{-1,-8}+x_{-1,-4}+x_{-1,1}+x_{-1,8}+x_{0,-8}+x_{0,-4}+x_{0,1}+x_{0,8},$$
$$s_1 = 2\big(x_{-1,-8}+x_{-1,-4}+x_{0,1}+x_{0,8}\big),$$
$$s_2 = 2\big(-x_{-1,-8}+2x_{-1,-4}+2x_{-1,1}-x_{-1,8}+x_{0,-8}+x_{0,8}\big),$$
$$s_3 = 4\big(-x_{-1,-8}+2x_{-1,-4}+x_{0,8}\big),$$
$$s_4 = 4\big(-3x_{-1,-8}+4x_{-1,-4}+4x_{-1,1}-3x_{-1,8}+x_{0,-8}+x_{0,8}\big),$$
$$s_5 = 8\big(x_{-1,-8}+x_{0,8}\big),$$
$$s_6 = 8\big(-x_{-1,-8}-x_{-1,8}+x_{0,-8}+x_{0,8}\big),$$
$$s_7 = 32.$$

Proof. The proof is immediate. We apply (1.1) and (1.2) again. $\square$

Corollary 1. *The congruence in the hypothesis of Theorem 5 is optimal if and only if*

$$x_{-1,-8} = a,$$
$$x_{-1,-4} = a+4e-2g,$$

$$
\begin{aligned}
x_{-1,1} &= -a + 8d - 4e + 2g - 2h,\\
x_{-1,8} &= -a + 16d - 4f - 2h,\\
x_{1,-8} &= a + 16d - 4f - 4g + 2h,\\
x_{1,-4} &= a + 32b - 16c - 40d + 4e + 8f + 2g + 2h,\\
x_{1,1} &= -a + 16c - 4e - 2g,\\
x_{1,8} &= -a + 4g,
\end{aligned}
$$

where $a, b, c, d, e, f, g, h \in \mathbb{C}_2$ are integers with odd a.

Proof. The congruence in the hypothesis of Theorem 5 is valid modulo $2^{\nu+5}$ if and only if s_0, s_1, s_2, s_3, s_4, s_5, s_6 satisfy (10.7) for some integers $b, c, d, e, f, g, h \in \mathbb{C}_2$. Taking $x_{-1,-8} = a$ we obtain a system of seven linear equations with seven unknowns $x_{-1,-4}$, $x_{-1,1}$, $x_{-1,8}$, $x_{0,-8}$, $x_{0,-4}$, $x_{0,1}$, $x_{0,8}$ and determinant -8. A standard computation gives the formulas of Corollary 1 at once. □

Corollary 2. *If the congruence in the hypothesis of Theorem 5 is optimal then all the $x_{-1,e}$, $x_{0,e}$ $(e \in \mathcal{T}_8)$ are odd. None of these coefficients can vanish in particular.*

12. The case $L = \{-1, 2\}$

In the case when $L = \{-1, 2\}$ we derive linear congruences among the conjectured orders of K_2-groups of the integers of quadratic fields. In this case the obtained congruence provides an analogue of the Gras and Uehara congruence in K_2-theory. Here $c(L) = 5$ and the congruences are valid modulo $2^{\nu+\lambda+1}$, where $\lambda \leq 5$.

Theorem 6. *Let $m > 1$ be an odd square-free integer having ν prime factors, and let $\Theta, \Psi : \mathbb{N} \to \mathbb{C}_2$ be multiplicative functions such that $\Psi(s) \equiv \Theta(s) \equiv 1 \pmod 2$ for any divisor $s \mid m$. In the notation of Theorem 3, for* *any 2-adic integers $x_{-1,e}$, $x_{2,e}$ $(e \in \mathcal{T}_8)$ not all even we have*

$$\begin{aligned}
&\sum_{e\in\mathcal{T}_8} x_{-1,e} \sum_{\substack{d\in\mathcal{T}_m\\ ed>1}} \Psi(|d|)\Big(\prod_{\substack{p\mid m\\ p \text{ prime}}} \big(1-\chi_{ed}(p)\Theta(p)p^2\big) \\
&\qquad\qquad\qquad\qquad - \delta_{d,1} \prod_{\substack{p\mid m\\ p \text{ prime}}} \big(1-\Theta(p)\big)\Big) K_2(ed) \\
&\quad + \sum_{e\in\mathcal{T}_8} x_{2,e} \sum_{\substack{d\in\mathcal{T}_m\\ ed<0}} \Psi(|d|)\Big(\prod_{\substack{p\mid m\\ p \text{ prime}}} \big(1-\chi_{ed}(p)\Theta(p)p^{-1}\big) \\
&\qquad\qquad\qquad\qquad - \delta_{d,1} \prod_{\substack{p\mid m\\ p \text{ prime}}} \big(1-\Theta(p)\big)\Big) K_2(ed), \\
&\quad + \frac{1}{6} x_{-1,1}\Big(\prod_{\substack{p\mid m\\ p \text{ prime}}} \big(1-\Theta(p)p^2\big) - \prod_{\substack{p\mid m\\ p \text{ prime}}} \big(1-\Theta(p)\big)\Big) \equiv 0 \pmod{2^{\nu+\lambda+1}},
\end{aligned}$$

where 2^λ is the greatest common divisor of the eight integers s_i $(0 \le i \le 7)$ defined by

$$\begin{aligned}
s_0 &= x_{-1,-8} + x_{-1,-4} + x_{-1,1} + x_{-1,8} + x_{2,-8} + x_{2,-4} + x_{2,1} + x_{2,8}, \\
s_1 &= 2\big(x_{-1,-8} + x_{-1,-4} + x_{2,1} + x_{2,8}\big), \\
s_2 &= 2\big(7x_{-1,-8} + 2x_{-1,-4} + 2x_{-1,1} + 7x_{-1,8} + 5x_{2,-8} \\
&\qquad + 4x_{2,-4} + 4x_{2,1} + 5x_{2,8}\big), \\
s_3 &= 4\big(-x_{-1,-8} + 2x_{-1,-4} + 4x_{2,1} + 5x_{2,8}\big), \\
s_4 &= 4\big(5x_{-1,-8} + 4x_{-1,-4} + 4x_{-1,1} + 5x_{-1,8} + x_{2,-8} + x_{2,8}\big), \\
s_5 &= 8\big(x_{-1,-8} + x_{2,8}\big), \\
s_6 &= 8\big(3x_{-1,-8} + 3x_{-1,8} + x_{2,-8} + x_{2,8}\big), \\
s_7 &= 32.
\end{aligned}$$

Proof. In order to obtain the above formulas for s_i, $0 \le i \le 6$ we make use of (1.1) and (1.2). $\square$

Corollary 1. *The congruence in the hypothesis of Theorem 6 is optimal if and only if*

$$\begin{aligned}
x_{-1,-8} &= a,\\
x_{-1,-4} &= -3a + 32c - 4e + 2g,\\
x_{-1,1} &= 3a + 64b - 32c - 8d + 4e - 2g + 2h,\\
x_{-1,8} &= -a - 128b + 16d + 4f - 6h,\\
x_{2,-8} &= a + 384b - 48d - 12f - 4g + 22h,\\
x_{2,-4} &= -3a - 288b + 16c + 40d - 4e + 8f + 6g - 18h,\\
x_{2,1} &= 3a - 16c + 4e - 6g,\\
x_{2,8} &= -a + 4g,
\end{aligned}$$

where $a, b, c, d, e, f, g, h \in \mathbb{C}_2$ are integers with a odd.

Proof. We proceed in the same way as in the proof of Corollary 1 to Theorem 5. Taking $x_{-1,-8} = a$ we obtain a system of seven linear equations with seven unknowns $x_{-1,-4}$, $x_{-1,1}$, $x_{-1,8}$, $x_{2,-8}$, $x_{2,-4}$, $x_{2,1}$, $x_{2,8}$ and determinant 8. An easy verification gives the above formulas immediately. □

Corollary 2. *If the congruence in the hypothesis of Theorem 6 is optimal then all the $x_{-1,e}$, $x_{2,e}$ ($e \in \mathcal{T}_8$) are odd. None of these coefficients can vanish in particular.*

13. The case $L = \{1, 2\}$

In the case when $L = \{1, 2\}$ we obtain linear congruences for class numbers of real quadratic fields and the orders of K_2-groups of the integers of imaginary quadratic fields. In this case $c(L) = 5$ and the obtained congruences are valid modulo $2^{\nu+\lambda+1}$, where $\lambda \leq 5$.

Theorem 7. *Let $m > 1$ be an odd square-free integer having ν prime factors, and let $\Theta, \Psi : \mathbb{N} \to \mathbb{C}_2$ be multiplicative functions such that $\Psi(s) \equiv$*

$\Theta(s) \equiv 1 \pmod 2$ *for any divisor* $s \mid m$*. In the notation of Theorem 3, for any 2-adic integers* $x_{1,e}$, $x_{2,e}$ $(e \in \mathcal{T}_8)$ *not all even we have*

$$\begin{aligned}
2\sum_{e\in\mathcal{T}_8} x_{1,e} \sum_{\substack{d\in\mathcal{T}_m\\ ed>1}} \Psi(|d|)\Big(\prod_{\substack{p\mid m\\ p\ \text{prime}}} \big(1-\chi_{ed}(p)\Theta(p)\big) \\
- \delta_{d,1} \prod_{\substack{p\mid m\\ p\ \text{prime}}} \big(1-\Theta(p)\big)\Big) H(ed) \\
+ \sum_{e\in\mathcal{T}_8} x_{2,e} \sum_{\substack{d\in\mathcal{T}_m\\ ed<0}} \Psi(|d|)\Big(\prod_{\substack{p\mid m\\ p\ \text{prime}}} \big(1-\chi_{ed}(p)\Theta(p)p^{-1}\big) \\
- \delta_{d,1} \prod_{\substack{p\mid m\\ p\ \text{prime}}} \big(1-\Theta(p)\big)\Big) K_2(ed), \\
- x_{1,1} \sum_{\substack{p\mid m\\ p\ \text{prime}}} \Theta(p)\log_2 p \prod_{\substack{q\mid (m/p)\\ q\ \text{prime}}} \big(1-\Theta(q)\big) \equiv 0 \pmod{2^{\nu+\lambda+1}},
\end{aligned}$$

where 2^λ *is the greatest common divisor of the eight integers* s_i $(0 \le i \le 7)$ *defined by*

$$s_0 = x_{1,-8} + x_{1,-4} + x_{1,1} + x_{1,8} + x_{2,-8} + x_{2,-4} + x_{2,1} + x_{2,8},$$

$$s_1 = 2\big(x_{1,-8} + x_{1,-4} + x_{2,1} + x_{2,8}\big),$$

$$s_2 = 2\big(3x_{1,-8} + 6x_{1,-4} + 6x_{1,1} + 3x_{1,8} + 5x_{2,-8} + 4x_{2,-4} + 4x_{2,1} + 5x_{2,8}\big),$$

$$s_3 = 4\big(x_{1,-8} + 2x_{1,-4} + 4x_{2,1} - x_{2,8}\big),$$

$$s_4 = 4\big(-3x_{1,-8} + 4x_{1,-4} + 4x_{1,1} - 3x_{1,8} + x_{2,-8} + x_{2,8}\big),$$

$$s_5 = 8\big(x_{1,-8} + x_{2,8}\big),$$

$$s_6 = 8\big(3x_{1,-8} + 3x_{1,8} + x_{2,-8} + x_{2,8}\big),$$

$$s_7 = 32.$$

PROOF. It follows from (1.1) and (1.2) that

$$\gcd(z_3, 32) = \gcd\big(4(3x_{1,-8} + 6x_{1,-4} + 4x_{2,1} + 5x_{2,8}), 32\big) = \gcd(s_3, 32)$$

and the corollary follows easily from Theorem 7. $\square$

Corollary 1. *The congruence in the hypothesis of Theorem 7 is optimal if and only if*

$$\begin{aligned}
x_{1,-8} &= a, \\
x_{1,-4} &= a + 32c - 4e - 10g, \\
x_{1,1} &= -a + 192b - 32c - 24d + 4e + 8f + 10g + 2h, \\
x_{1,8} &= -a + 128b - 16d + 4f + 2h, \\
x_{2,-8} &= a - 384b + 48d - 12f - 4g - 2h, \\
x_{2,-4} &= a + 96b + 16c - 8d - 4e - 6g - 2h, \\
x_{2,1} &= -a - 16c + 4e + 6g, \\
x_{2,8} &= -a + 4g,
\end{aligned}$$

where $a, b, c, d, e, f, g, h \in \mathbb{C}_2$ are integers with a odd.

Proof. The proof is standard. We proceed in the same way as in the proof of Corollary 1 to Theorem 5. Taking $x_{1,-8} = a$ we obtain a system of seven linear equations with seven unknowns $x_{-1,-4}$, $x_{1,1}$, $x_{1,8}$, $x_{2,-8}$, $x_{2,-4}$, $x_{2,1}$, $x_{2,8}$ and determinant -8. The details are left to the reader. $\square$

Corollary 2. *If the congruence in the hypothesis of Theorem 7 is optimal then all the $x_{1,e}$, $x_{2,e}$ $(e \in \mathcal{T}_8)$ are odd. None of these coefficients can vanish in particular.*

14. The cases $L = \{-1, 1\}$ and $L = \{0, 2\}$

In the case when $L = \{-1, 1\}$ (resp. $L = \{0, 2\}$) we obtain linear congruences between class numbers and the orders of K_2-groups of the integers of real (resp. imaginary) quadratic fields. In both the cases $c(L) = 6$ and the obtained congruences are valid modulo $2^{\nu+\lambda+1}$, where $\lambda \leq 6$.

Theorem 8. *Let $m > 1$ be an odd square-free integer having ν prime factors, and let $\Theta, \Psi : \mathbb{N} \to \mathbb{C}_2$ be multiplicative functions such that $\Psi(s) \equiv \Theta(s) \equiv 1 \pmod 2$ for any divisor $s \mid m$. In the notation of Theorem 3, for any 2-adic integers $x_{-1,e}$, $x_{1,e}$ $(e \in \mathcal{T}_8)$ not all even we have*

$$\begin{aligned}
&\sum_{e\in\mathcal{T}_8} x_{-1,e} \sum_{\substack{d\in\mathcal{T}_m\\ ed>1}} \Psi(|d|)\Big(\prod_{\substack{p\mid m\\ p \text{ prime}}} \big(1-\chi_{ed}(p)\Theta(p)p^2\big) \\
&\qquad\qquad - \delta_{d,1} \prod_{\substack{p\mid m\\ p \text{ prime}}} \big(1-\Theta(p)\big)\Big) K_2(ed) \\
&+ 2\sum_{e\in\mathcal{T}_8} x_{1,e} \sum_{\substack{d\in\mathcal{T}_m\\ ed>1}} \Psi(|d|)\Big(\prod_{\substack{p\mid m\\ p \text{ prime}}} \big(1-\chi_{ed}(p)\Theta(p)\big) \\
&\qquad\qquad - \delta_{d,1} \prod_{\substack{p\mid m\\ p \text{ prime}}} \big(1-\Theta(p)\big)\Big) H(ed) \\
&+ \frac{1}{6} x_{-1,1}\Big(\prod_{\substack{p\mid m\\ p \text{ prime}}} \big(1-\Theta(p)p^2\big) - \prod_{\substack{p\mid m\\ p \text{ prime}}} \big(1-\Theta(p)\big)\Big) \\
&- x_{1,1} \sum_{\substack{p\mid m\\ p \text{ prime}}} \Theta(p)\log_2 p \prod_{\substack{q\mid (m/p)\\ q \text{ prime}}} \big(1-\Theta(q)\big) \equiv 0 \pmod{2^{\nu+\lambda+1}},
\end{aligned}$$

where 2^λ is the greatest common divisor of the eight integers s_i $(0 \le i \le 7)$ defined by

$$s_0 = x_{-1,-8} + x_{-1,-4} + x_{-1,1} + x_{-1,8} + x_{1,-8} + x_{1,-4} + x_{1,1} + x_{1,8},$$

$$s_1 = 2\big(x_{-1,-8} + x_{-1,-4} + x_{1,-8} + x_{1,-4}\big),$$

$$\begin{aligned} s_2 = 2\big(&-3x_{-1,-8} + 6x_{-1,-4} + 6x_{-1,1} - 3x_{-1,8} \\ &+ x_{1,-8} + 2x_{1,-4} + 2x_{1,1} + x_{1,8}\big), \end{aligned}$$

$$s_3 = 4\big(-3x_{-1,-8} + 6x_{-1,-4} + x_{1,-8} + 2x_{1,-4}\big),$$

$$\begin{aligned} s_4 = 4\big(&5x_{-1,-8} + 4x_{-1,-4} + 4x_{-1,1} + 5x_{-1,8} + 5x_{1,-8} \\ &+ 4x_{1,-4} + 4x_{1,1} + 5x_{1,8}\big), \end{aligned}$$

$$s_5 = 8\big(5x_{-1,-8} + 4x_{-1,-4} + 5x_{1,-8} + 4x_{1,-4}\big),$$

$$s_6 = 8\big(3x_{-1,-8} + 3x_{-1,8} - x_{1,-8} - x_{1,8}\big),$$

$$s_7 = 64.$$

Proof. Note that in the case when $L = \{-1, 1\}$ we have

$$z_8 \equiv -2z_6 \pmod{64}, \quad z_7 \equiv -2z_5 \pmod{64},$$

and in consequence we may ignore z_8 and z_7 (the z_n with $n = 2c(L) - 4$, $2c(L) - 5$). In order to obtain formulas for s_i, $0 \le i \le 6$ we use (1.1) and (1.2). For example, we have

$$\begin{aligned}\gcd(z_3, 64) &= \gcd\big(4(-9x_{-1,-8} + 2x_{-1,-4} + 3x_{1,-8} + 6x_{1,-4}), 64\big)\\ &= \gcd(s_3, 64).\end{aligned}$$

The corollary follows easily from Theorem 8. □

Corollary 1. *The congruence in the hypothesis of Theorem 8 is optimal if and only if*

$$\begin{aligned}
x_{-1,-8} &= a,\\
x_{-1,-4} &= a - 48c + 4e + 2g,\\
x_{-1,1} &= -a - 160b + 48c + 8d - 4e + 8f - 2g + 2h,\\
x_{-1,8} &= -a - 64b + 4f + 2h,\\
x_{1,-8} &= -a - 128c + 8g,\\
x_{1,-4} &= -a + 208c - 4e - 10g,\\
x_{1,1} &= a + 480b - 208c - 8d + 4e - 24f + 10g - 2h,\\
x_{1,8} &= a - 192b + 128c + 12f - 8g - 2h,
\end{aligned}$$

where $a, b, c, d, e, f, g, h \in \mathbb{C}_2$ are integers with a odd.

Proof. The congruence in the hypothesis of Theorem 8 is valid modulo $2^{\nu+6}$ if and only if

$$\begin{gathered} s_0 = 64b,\ s_1 = 64c,\ s_2 = 64d,\ s_3 = 64e,\\ s_4 = 64f,\ s_5 = 64g,\ s_6 = 64h \end{gathered} \tag{14.8}$$

for some integers $b, c, d, e, f, g, h \in \mathbb{C}_2$. Taking $x_{0,-8} = a$ we obtain a system of seven linear equations with seven unknowns $x_{-1,-4}$, $x_{-1,1}$, $x_{-1,8}$, $x_{1,-8}$, $x_{1,-4}$, $x_{1,1}$, $x_{1,8}$ and determinant -64. A standard computation gives the formulas of Corollary 1 at once. □

Corollary 2. *If the congruence in the hypothesis of Theorem 8 is optimal then all the $x_{k,e}$ are odd. None of these coefficients can vanish in particular.*

Theorem 9. *Let $m > 1$ be an odd square-free integer having ν prime factors, and let $\Theta, \Psi : \mathbb{N} \to \mathbb{C}_2$ be multiplicative functions such that $\Psi(s) \equiv \Theta(s) \equiv 1 \pmod 2$ for any divisor $s \mid m$. In the notation of Theorem 3, for any 2-adic integers $x_{0,e}$, $x_{2,e}$ ($e \in \mathcal{T}_8$) not all even we have*

$$\begin{aligned}
2\sum_{e\in\mathcal{T}_8} x_{0,e} \sum_{\substack{d\in\mathcal{T}_m\\ ed<0}} \Psi(|d|)\Big(&\prod_{\substack{p\mid m\\ p \text{ prime}}} \big(1-\chi_{ed}(p)\Theta(p)p\big) \\
&- \delta_{d,1} \prod_{\substack{p\mid m\\ p \text{ prime}}} \big(1-\Theta(p)\big)\Big) H(ed) \\
+ \sum_{e\in\mathcal{T}_8} x_{2,e} \sum_{\substack{d\in\mathcal{T}_m\\ ed<0}} \Psi(|d|)\Big(&\prod_{\substack{p\mid m\\ p \text{ prime}}} \big(1-\chi_{ed}(p)\Theta(p)p^{-1}\big) \\
&- \delta_{d,1} \prod_{\substack{p\mid m\\ p \text{ prime}}} \big(1-\Theta(p)\big)\Big) K_2(ed), \\
\equiv 0 \pmod{2^{\nu+\lambda+1}}&,
\end{aligned}$$

where 2^λ is the greatest common divisor of the eight integers s_i $(0 \le i \le 7)$ defined by

$$s_0 = x_{0,-8} + x_{0,-4} + x_{0,1} + x_{0,8} + x_{2,-8} + x_{2,-4} + x_{2,1} + x_{2,8},$$

$$s_1 = 2\big(x_{0,1} + x_{0,8} + x_{2,1} + x_{2,8}\big),$$

$$s_2 = 2\big(9x_{0,-8} + 9x_{0,8} + 5x_{2,-8} + 4x_{2,-4} + 4x_{2,1} + 5x_{2,8}\big),$$

$$s_3 = 4\big(9x_{0,8} + 4x_{2,1} + 5x_{2,8}\big),$$

$$s_4 = 4\big(x_{0,-8} + x_{0,8} + x_{2,-8} + x_{2,8}\big),$$

$$s_5 = 8\big(x_{0,8} + x_{2,8}\big),$$

$$s_6 = 8\big(5x_{0,-8} + 5x_{0,8} + x_{2,-8} + x_{2,8}\big),$$

$$s_7 = 64.$$

Proof. Note that in the case when $L = \{0,2\}$ we have

$$z_8 \equiv 2z_6 \pmod{64}, \quad z_7 \equiv 2z_5 \pmod{64},$$

and in consequence we may ignore the z_8 and z_7 (the z_n with $n = 2c(L)-4$, $2c(L)-5$). We apply (1.1) and (1.2) again. □

Corollary 1. *The congruence in the hypothesis of Theorem 9 is optimal if and only if*

$$\begin{aligned}
x_{0,-8} &= a,\\
x_{0,-4} &= a + 64b - 32c - 8d + 4e + 4f - 2g,\\
x_{0,1} &= -a + 32c - 4e - 4f + 2g + 2h,\\
x_{0,8} &= -a - 4f + 2h,\\
x_{2,-8} &= -a + 16f - 8g,\\
x_{2,-4} &= -a + 8d - 4e - 20f + 10g,\\
x_{2,1} &= a + 4e + 4f - 10g - 2h,\\
x_{2,8} &= a + 4f + 8g - 2h,
\end{aligned}$$

where $a,b,c,d,e,f,g,h \in \mathbb{C}_2$ are integers with a odd.

Proof. The congruence in the hypothesis of Theorem 9 is valid modulo $2^{\nu+6}$ if and only if s_0, s_1, s_2, s_3, s_4, s_5, s_6 satisfy (14.8) for some integers $b,c,d,e,f,g,h \in \mathbb{C}_2$. Taking $x_{0,-8} = a$ we obtain a system of seven linear equations with seven unknowns $x_{0,-4}$, $x_{0,1}$, $x_{0,8}$, $x_{2,-8}$, $x_{2,-4}$, $x_{2,1}$, $x_{2,8}$ and determinant 64. An easy verification gives the formulas of Corollary 1 at once. □

Corollary 2. *If the congruence in the hypothesis of Theorem 9 is optimal then all the $x_{0,e}$, $x_{2,e}$ ($e \in \mathcal{T}_8$) are odd. None of these coefficients can vanish in particular.*

15. Concluding remarks

Uehara's approach used in [8] and [10] gives a method of producing linear congruences. It would be interesting to use this method to find for given λ explicit formulas for the $x_{k,e}$ such that the linear congruences are valid modulo $2^{\nu+\lambda}$. This approach should yield many new congruences between class numbers and the orders of K_2-groups of the rings of integers of quadratic fields. In the case of the orders of K_2-groups for imaginary quadratic fields such congruences would be completely new. The detailed results will appear in forthcoming publications.

Another direction for further investigation would be to extend Wójcik's congruence [10] by giving a congruence for a linear combination of the values $L_2(k,\chi\omega^{1-k})$, where the numbers k are taken from any finite subset of the integers. Wójcik's congruence involved the case when this subset consisted of consecutive integers. Urbanowicz and Wójcik [8] found such a congruence for any subset of the set $\{-1,0,1,2\}$.

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