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On special weakly symmetric Riemannian manifolds

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Abstract. In this paper we have defined a special weakly conharmonic symmetric Riemannian manifold $(SWNS)n$ and special weakly Ricci symmetric Riemannian manifold $(SWRS)n$ and have investigated some results.

1. Introduction

The notions of weakly symmetric and weakly projective symmetric Riemannian manifold have recently been introduced and studied by L. Tamássy and T. Q. Binh ([1], [2]).

Let M be an n-dimensional Riemannian manifold and $\mathfrak{X}(M)$ denote the set of differentiable vector fields on M. Let $D_X Y$ denote the covariant derivative of Y with respect to X and $K(X,Y,Z)$ be the Riemannian curvature tensor for $X,Y,Z \in \mathfrak{X}(M)$. Let us consider the relation

$$(1.1)\qquad \begin{aligned}(D_X K)(Y,Z,V) = {}&\omega(X)K(Y,Z,V) + \beta(Y)K(X,Z,V)\\ &+ \gamma(Z)K(Y,X,V) + \sigma(V)K(Y,Z,X)\\ &+ \langle K(Y,Z,V),X\rangle F,\end{aligned}$$

where ω, β, γ and σ are non-zero 1-forms; F is a vector field. Such an n-dimensional Riemannian manifold is called a weakly symmetric Riemannian manifold and is denoted by $(WS)n$. If $\beta = \gamma = \sigma = \frac{1}{2}\omega$ and $F = \alpha$ in (1.1) then the manifold reduces to pseudo symmetric manifold according

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to Chaki [4] and if $\omega = \beta = \gamma = \sigma = 0$ and $F = 0$, then the manifold reduces to symmetric manifold according to Kobayashi and Nomizu [3].

A Riemannian manifold M is said to be weakly Ricci symmetric Riemannian manifold [2] and is denoted by $(WRS)n$, if there exist 1-forms ρ, μ, ν such that

$$(D_X \operatorname{Ric})(Y,Z) = \rho(X)\operatorname{Ric}(Y,Z) + \mu(Y)\operatorname{Ric}(X,Z) + \nu(Z)\operatorname{Ric}(Y,X). \tag{1.2}$$

If $\rho = \mu = \nu$ in (1.2), then the manifold reduces to pseudo Ricci Symmetric manifold according to Chaki [6] and if $\rho = \mu = \nu = 0$ in (1.2), then $(WRS)n$ reduces to Ricci Symmetric manifold.

An n-dimensional Riemannian manifold in which the conharmonic curvature tensor $N(X,Y,Z)$ satisfies the condition

$$\begin{aligned}(D_X N)(Y,Z,V) = {} & \omega(X)N(Y,Z,V) + \beta(Y)N(X,Z,V) \\ & + \gamma(Z)N(Y,X,V) + \sigma(V)N(Y,Z,X) \\ & + \langle N(Y,Z,V),X\rangle F,\end{aligned} \tag{1.3}$$

where ω, β, γ and σ are non-zero 1-forms; F is a vector field and $N(X,Y,Z)$ is defined by [7]

$$\begin{aligned}N(X,Y,Z) = K(X,Y,Z) - \frac{1}{n-2}\big[& \operatorname{Ric}(Y,Z)X - \operatorname{Ric}(X,Z)Y \\ & + g(Y,Z)R(X) - g(X,Z)R(Y)\big],\end{aligned} \tag{1.4}$$

is called a weakly conharmonically symmetric manifold $(WNS)n$.

Let

$$'N(X,Y,Z,V) = g(N(X,Y,Z),V). \tag{1.5}$$

Then from (1.4), we get

$$\begin{aligned}'N(X,Y,Z,V) = {}'K(X,Y,Z,V) - \frac{1}{n-2}\big[& \operatorname{Ric}(Y,Z)g(X,V) \\ & - \operatorname{Ric}(X,Z)g(Y,V) + g(Y,Z)\operatorname{Ric}(X,V) \\ & - g(X,Z)\operatorname{Ric}(Y,V)\big],\end{aligned} \tag{1.6}$$

where

$$'K(X,Y,Z,V) = g(K(X,Y,Z),V). \tag{1.7}$$

Let

$$h(X,V) = {}'N(X,e_i,e_i,V), \tag{1.8}$$

then from (1.6), we have

$$h(X,V) = \frac{n}{n-2}\,\mathrm{Ric}(X,V) - \frac{r}{n-2}g(X,V), \tag{1.9}$$

where r is the scalar curvature.

The conformal curvature tensor $C(X,Y,Z)$ and the projective curvature tensor $P(X,Y,Z)$ are given by [5]

$$\begin{aligned} C(X,Y,Z) = K(X,Y,Z) &- \frac{1}{n-2}\big[\,\mathrm{Ric}(Y,Z)X - \mathrm{Ric}(X,Z)Y \\ &+ g(Y,Z)R(X) - g(X,Z)R(Y)\big] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{1.10}$$

and

$$P(X,Y,Z) = K(X,Y,Z) - \frac{1}{n-1}[\mathrm{Ric}(Y,Z)X - \mathrm{Ric}(X,Z)Y], \tag{1.11}$$

respectively.

If a Riemannian manifold is an Einstein manifold, then [5]

$$\mathrm{Ric}(X,Y) = kg(X,Y), \tag{1.12}$$

where k is constant. From (1.12), we have

$$R(X) = kX. \tag{1.13}$$

Contracting (1.1), we get

$$r = nk. \tag{1.14}$$

2. Special weakly conharmonically symmetric Riemannian manifold

A weakly symmetric Riemannian manifold $(WS)n$ of Tamássy and Binh [2] is a locally symmetric Riemannian manifold if (i) $\omega = \beta = \sigma = 0$

and (ii) $F = 0$ hold in (1.1). A $(WS)n$ is special if the 1-forms ω, β, γ, σ and the vector field F satisfy some special conditions, but do not vanish simultaneously.

Analogously we can give the following

Definition 2.1. Let $\frac{1}{2}\omega = \beta = \gamma = \sigma = \alpha$ and $F = 0$. Then (1.1) reduces to the form

$$\begin{aligned}(2.1)\qquad (D_X K)(Y,Z,V) &= 2\alpha(X)K(Y,Z,V) + \alpha(Y)K(X,Z,V)\\ &\quad + \alpha(Z)K(Y,X,V) + \alpha(V)K(Y,Z,X),\end{aligned}$$

where α is a non-zero 1-form and is defined as

$$(2.2)\qquad \alpha(X) = g(X,P),\ \forall X,$$

where P is a vector field. Such an n-dimensional Riemannian manifold is a special weakly symmetric Riemannian manifold and we write it as $(SWS)n$. If we replace K by N in (2.1), then it reduces to

$$\begin{aligned}(2.3)\qquad (D_X N)(Y,Z,V) &= 2\alpha(X)N(Y,Z,V) + \alpha(Y)N(X,Z,V)\\ &\quad + \alpha(Z)N(Y,X,V) + \alpha(V)N(Y,Z,X).\end{aligned}$$

A manifold satisfying the condition (2.3) is called a special weakly conharmonically symmetric Riemannian manifold and is denoted by $(SWNS)n$.

We consider a $(SWNS)n$. Taking covariant derivative of (1.4) with respect to X and then using (2.3), we get

$$\begin{aligned}(2.4)\qquad & \alpha(X)(Y,Z,V) + \alpha(Y)N(X,Z,V) + \alpha(Z)N(Y,X,V)\\ &\quad + \alpha(V)N(Y,Z,X) = (D_X K)(Y,Z,V)\\ &\quad - \frac{1}{n-1}\big[(D_X \operatorname{Ric})(Z,V)Y - (D_X \operatorname{Ric})(Y,V)Z\\ &\quad + g(Z,V)(D_X R)(Y) - g(Y,V)(D_X R)(Z)\big].\end{aligned}$$

By virtue of (1.4), the equation (2.4) reduces to

$$\begin{aligned}(2.5)\qquad & (D_X K)(Y,Z,V) - 2\alpha(X)K(Y,Z,V) - \alpha(Y)K(X,Z,V)\\ & - \alpha(Z)K(Y,X,V) - \alpha(V)K(Y,Z,X)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n-2}\big[(D_X \operatorname{Ric})(Z,V)Y - (D_X \operatorname{Ric})(Y,V)Z \\
&+ g(Z,V)(D_X R)(Y) - g(Y,V)(D_X R)(Z) \\
&- 2\alpha(X)\{\operatorname{Ric}(Z,V)Y - \operatorname{Ric}(Y,V)Z + g(Z,V)R(Y) - g(Y,V)R(Z)\} \\
&- \alpha(Y)\{\operatorname{Ric}(Z,V)X - \operatorname{Ric}(X,V)Z + g(Z,V)R(X) - g(X,V)R(Z)\} \\
&- \alpha(Z)\{\operatorname{Ric}(X,V)Y - \operatorname{Ric}(Y,V)X + g(X,V)R(Y) - g(Y,V)R(X)\} \\
&- \alpha(V)\{\operatorname{Ric}(Z,X)Y - \operatorname{Ric}(Y,X)Z + g(Z,X)R(Y) - g(Y,X)R(Z)\}\big].
\end{aligned}
$$

Permuting equation (2.5) twice with respect to X, Y, Z; adding the three obtained equations and using Bianchi's first and second identities, we have

$$
\begin{aligned}
(2.6)\qquad & 2\alpha(X)K(Y,Z,V) + 2\alpha(Y)K(Z,X,V) + 2\alpha(Z)K(X,Y,V) \\
&+ \alpha(Y)K(X,Z,V) + \alpha(Z)K(Y,X,V) + \alpha(X)K(Z,Y,V) \\
&+ \alpha(Z)K(Y,X,V) + \alpha(X)K(Z,Y,V) + \alpha(Y)K(X,Z,V) \\
&+ \frac{1}{n-2}\big[(D_X \operatorname{Ric})(Z,V)Y + (D_Y \operatorname{Ric})(X,V)Z + (D_Z \operatorname{Ric})(Y,V)X \\
&- (D_X \operatorname{Ric})(Y,V)Z - (D_Y \operatorname{Ric})(Z,V)X - (D_Z \operatorname{Ric})(X,V)Y \\
&+ g(Z,V)(D_X R)(Y) + g(X,V)(D_Y, R)(Z) + g(Y,V)(D_Z R)(X) \\
&- g(Y,V)(D_X R)(Z) - g(Z,V)(D_Y R)(X) - g(X,V)(D_Z R)(Y) \\
&-2\alpha(X)\{\operatorname{Ric}(Z,V)Y - \operatorname{Ric}(Y,V)Z + g(Z,V)R(Y) - g(Y,V)R(Z)\} \\
&-2\alpha(Y)\{\operatorname{Ric}(X,V)Z - \operatorname{Ric}(Z,V)X + g(X,V)R(Z) - g(Z,V)R(X)\} \\
&-2\alpha(Z)\{\operatorname{Ric}(Y,V)X - \operatorname{Ric}(X,V)Y + g(Y,V)R(X) - g(X,V)R(Y)\} \\
&-\alpha(Y)\{\operatorname{Ric}(Z,V)X - \operatorname{Ric}(X,V)Z + g(Z,V)R(X) - g(X,V)R(Z)\} \\
&-\alpha(Z)\{\operatorname{Ric}(X,V)Y - \operatorname{Ric}(Y,V)X + g(X,V)R(Y) - g(Y,V)R(X)\} \\
&-\alpha(X)\{\operatorname{Ric}(Y,V)Z - \operatorname{Ric}(Z,V)Y + g(Y,V)R(Z) - g(Z,V)R(Y)\} \\
&-\alpha(Z)\{\operatorname{Ric}(X,V)Y - \operatorname{Ric}(Y,V)X + g(X,V)R(Y) - g(Y,V)R(X)\} \\
&-\alpha(X)\{\operatorname{Ric}(Y,V)Z - \operatorname{Ric}(Z,V)Y + g(Y,V)R(Z) - g(Z,V)R(Y)\} \\
&-\alpha(Y)\{\operatorname{Ric}(Z,V)X - \operatorname{Ric}(X,V)Z + g(Z,V)R(X) - g(X,V)R(Z)\} \\
&-\alpha(V)\{\operatorname{Ric}(Z,X)Y - \operatorname{Ric}(Y,X)Z + g(Z,X)R(Y) - g(Y,X)R(Z)\} \\
&+ \operatorname{Ric}(X,Y)Z - \operatorname{Ric}(Z,Y)X + g(X,Y)R(Z) - g(Z,Y)R(X) \\
&+ \operatorname{Ric}(Y,Z)X - \operatorname{Ric}(X,Z)Y + g(Y,Z)R(X) - g(X,Z)R(Y)\big] = 0.
\end{aligned}
$$

Using symmetric properties of Ricci tensor and the skew-symmetric properties of curvature tensor in (2.6), we get

$$\begin{aligned}(D_X \operatorname{Ric})(Z,V)Y + (D_Y \operatorname{Ric})(X,V)Z + (D_Z \operatorname{Ric})(Y,V)X \\ - (D_X \operatorname{Ric})(Y,V)Z - (D_Y \operatorname{Ric})(Z,V)X - (D_Z \operatorname{Ric})(X,V)Y \\ + g(Z,V)(D_X R)(Y) + g(X,V)(D_Y R)(Z) \\ + g(Y,V)(D_Z R)(X) - g(Y,V)(D_X R)(Z) \\ - g(Z,V)(D_Y R)(X) - g(X,V)(D_Z R)(Y) = 0.\end{aligned} \tag{2.7}$$

Contracting(2.7) with respect to X, we get

$$\begin{aligned}(D_Y \operatorname{Ric})(Z,V) + (D_Y \operatorname{Ric})(Z,V) + n(D_Z \operatorname{Ric})(Y,V) - (D_Z \operatorname{Ric})(Y,V) \\ -n(D_Y \operatorname{Ric})(Z,V) - (D_Z \operatorname{Ric})(Y,V) + g(Z,V)\left(\frac{1}{2}Yr\right) \\ +g\big((D_Y R)(Z),V\big) + g(Y,V)(Zr) - g(Y,V)\left(\frac{1}{2}Zr\right) \\ -g(Z,V)(Yr) - g\big((D_Z R)(Y),V\big) = 0\end{aligned}$$

or,

$$\begin{aligned}(n-2)(D_Z \operatorname{Ric})(Y,V) - (n-2)(D_Y \operatorname{Ric})(Z,V) \\ + g\big((D_Y R)(Z),V\big) - g\big((D_Z R)(Y)V\big) \\ + \frac{1}{2}g(Y,V)(Zr) - \frac{1}{2}g(Z,V)(Yr) = 0.\end{aligned} \tag{2.8}$$

Factoring off V in (2.8), we get

$$\begin{aligned}(n-2)(D_Z R)(Y) - (n-2)(D_Y R)(Z) + (D_Y R)(Z) - (D_Z R)(Y) \\ +\frac{1}{2}(YZr) - \frac{1}{2}(ZYr) = 0\end{aligned}$$

or

$$(D_Z R)(Y) - (D_Y R)(Z) = 0. \tag{2.9}$$

Contracting (2.9) with respect to Y, we get

$$Zr = 0,$$

which shows that the scalar curvature r is constant.

This leads us to the following

Theorem 1. *In a* $(SWNS)n$*, given by* (2.3) *the scalar curvature* r *must be constant.*

Now let M be a $(SWNS)n$ and let it admit a unit parallel vector field V, that is

$$D_X V = 0. \tag{2.10}$$

Applying Ricci identity to (2.10), we get

$$K(X,Y,V) = 0 \tag{2.11}$$

or,

$$'K(X,Y,Z,V) = 0, \tag{2.12}$$

and therefore

$$\mathrm{Ric}(X,V) = 0. \tag{2.13}$$

Using (2.12) and (2.13) in (1.6), we get

$$'N(X,Y,Z,V) = 0. \tag{2.14}$$

Using (1.8) in (2.14), we get

$$h(X,V) = 0. \tag{2.15}$$

Taking an account of (2.15) and the fact that V is a unit parallel vector field it follows from (1.9) that

$$r = 0. \tag{2.16}$$

Now from (1.8) and (2.3), we have

$$\begin{aligned}(D_Z h)(X,V) = (D_Z{}'N)(X,e_i,e_i,V) &= 2\alpha(Z)'N(X,e_i,e_i,V) \\ &+ \alpha(X)'N(Z,e_i,e_i,V) + \alpha(e_i)'N(X,Z,e_i,V) \\ &+ \alpha(e_i)'N(X,e_i,Z,V) + \alpha(V)'N(X,e_i,e_i,Z).\end{aligned} \tag{2.17}$$

Using (1.6), (2.10), (2.13), (2.15) and (2.16) the above equation takes the form

$$\alpha(V)\,\mathrm{Ric}(X,Z) = 0. \tag{2.18}$$

Since $\alpha(V) \neq 0$, it follows from (2.18) that

$$\mathrm{Ric}(X, Z) = 0 \tag{2.19}$$

or,

$$R(X) = 0. \tag{2.20}$$

By virtue of equations (2.19) and (2.20) the equation (1.4) gives

$$N(X, Y, Z) = K(X, Y, Z). \tag{2.21}$$

But by virtue of (1.3) and (2.21), the relation (1.1) holds, i.e. a weakly conharmonically symmetric $(WNS)n$ reduces to a weakly symmetric manifold.

Thus we have the following

Theorem 2. *If a special weakly conharmonically symmetric manifold admits a unit parallel vector field, then it is a weakly symmetric manifold.*

By virtue of (1.12) and (1.13), the equation (1.4) reduces to

$$N(Y, Z, V) = K(Y, Z, V) - \frac{2k}{n-2}[g(Z, V) - g(Y, V)Z]. \tag{2.22}$$

Taking covariant derivative of (2.22) with respect to X, we get

$$(D_X N)(Y, Z, V) = (D_X K)(Y, Z, V). \tag{2.23}$$

By virtue of (2.22) and (2.23), the equation (2.3) reduces to the form

$$\begin{aligned}(D_X K)(Y, Z, V) = {} & 2\alpha(X)\big[K(Y, Z, V) \\ & - \frac{2k}{n-2}\{g(Z, V)Y - g(Y, V)Z\}\big] \\ & + \alpha(Y)\big[K(X, Z, V) - \frac{2k}{n-2}\{g(Z, V)X - g(X, V)Z\}\big] \\ & + \alpha(Z)\big[K(Y, X, V) - \frac{2k}{n-2}\{g(X, V)Y - g(Y, V)X\}\big] \\ & + \alpha(V)\big[K(Y, Z, X) - \frac{2k}{n-2}\{g(Z, X)Y - g(Y, X)Z\}\big].\end{aligned} \tag{2.24}$$

Thus, we have the following

Theorem 3. *The necessary and sufficient condition for an Einstein special weakly conharmonically symmetric manifold* $(SWNS)n$ *to be a special weakly symmetric manifold* $(SWS)n$ *is that*

$$\big[\{2\alpha(X)Y+\alpha(Y)X\}g(Z,V)-\{2\alpha(X)Z+\alpha(Z)X\}g(Y,V)$$
$$+\{\alpha(Z)Y-\alpha(Y)Z\}g(X,V)+\alpha(V)g(Z,X)Y-\alpha(V)g(Y,X)Z\big]=0.$$

Let m be an arbitrary point of an n-dimensional Riemannian manifold M and $e_i \in \mathfrak{X}(M)$ $(i=1,2,\dots,n)$ an orthonormal and parallel vector system around m. We consider at m the following relation

$$\sum_{i=1}^{n}(D_X{'K})(e_i,Z,V,e_i)=\sum_{i=1}^{n}D_X{'K}(e_i,Z,V,e_i) \tag{2.25}$$
$$-\sum_{i=1}^{n}{'K}(D_Xe_i,Z,V,e_i)-\sum_{i=1}^{n}K(e_i,D_XZ,V,e_i)-\sum_{i=1}^{n}{'K}(e_i,Z,D_XV,e_i).$$

Since we have

$$\sum_{i=1}^{n}{'K}(e_i,Z,V,e_i)=\mathrm{Ric}(Z,V)=\mathrm{Ric}(V,Z) \tag{2.26}$$

and $D_Xe_i \mid m=0$, we obtain

$$\sum_{i=1}^{n}(D_X{'K})(e_i,Z,V,e_i)=D_X\,\mathrm{Ric}(Z,V)-\mathrm{Ric}(D_XZ,V) \tag{2.27}$$
$$-\mathrm{Ric}(Z,D_XV)=(D_X\,\mathrm{Ric})(Z,V).$$

On the other hand we have from (2.1)

$$\sum_{i=1}^{n}(D_X{'K})(e_i,Z,V,e_i)=\sum_{i=1}^{n}\big[2\alpha(X){'K}(e_i,Z,V,e_i) \tag{2.28}$$
$$+\alpha(e_i){'K}(X,Z,V,e_i)+\alpha(Z){'K}(e_i,X,V,e_i)$$
$$+\alpha(V)K(e_i,Z,X,e_i)\big].$$

Using (2.26) and equating the right hand sides of (2.27) and (2.28), we obtain

$$(D_X \operatorname{Ric})(Z,V) = 2\alpha(X)\operatorname{Ric}(Z,V) + \alpha(K(X,Z,V)) + \alpha(Z)\operatorname{Ric}(X,V) + \alpha(V)\operatorname{Ric}(Z,X). \tag{2.29}$$

Now we assume that the n-dimensional ($n > 2$) Riemannian manifold M is an Einstein manofold, i.e.

$$\operatorname{Ric} = kg \tag{2.30}$$

with nonvanishing constant k. Then we have $D_X \operatorname{Ric} = kD_X g = 0$. Taking $Z = V = e_i$ in (2.29) and performing a summation over i, we get

$$0 = \sum_{i=1}^{n} \big[2\alpha(X)\operatorname{Ric}(e_i,e_i) + \langle K(X,e_i,e_i),B\rangle + \alpha(e_i)\operatorname{Ric}(X,e_i) + \alpha(e_i)\operatorname{Ric}(e_i,X)\big],$$

where $B \in \mathfrak{X}(M)$ is the vector field corresponding to α given by

$$\alpha(X) = \langle X,B\rangle$$

for all X. Then by (2.26) and (2.30), we have

$$0 = nc2\alpha(X) + \operatorname{Ric}(X,B) + \alpha(e_i)c\langle X,e_i\rangle + \alpha(e_i)c\langle X,e_i\rangle,$$

where $c = \operatorname{Ric}(e_i,e_i)$ and $c\langle X,e_i\rangle = \operatorname{Ric}(X,e_i)$ or,

$$0 = c\big[2n\alpha(X) + \alpha(X) + \alpha(X) + \alpha(X)\big],$$

which, in virtue of $c \neq 0$, reduces to

$$(2n+3)\alpha(X) = 0, \quad \text{for all } X.$$

Thus, we have the following

Theorem 4. *If a special weakly symmetric Riemannian manifold $(SWS)n$ is an Einstein manifold then the 1-form α must vanish.*

3. Special weakly Ricci symmetric Riemannian manifold

According to Tamássy and Binh [2], a weakly Ricci symmetric Riemannian manifold $(WRS)n$ is locally Ricci symmetric Riemannian manifold if $\rho = \mu = \nu = 0$ in (1.2). A $(WRS)n$ is called special if ρ, μ, ν satisfy some special conditions but do not vanish simultaneously.

Analogously we can give the following

Definition 3.1. Let $\frac{1}{2}\rho = \mu = \nu = \alpha$, then (1.2) reduces to the form

$$(D_X \operatorname{Ric})(Y,Z) = 2\alpha(X)\operatorname{Ric}(Y,Z) + \alpha(Y)\operatorname{Ric}(X,Z) + \alpha(Z)\operatorname{Ric}(Y,X), \tag{3.1}$$

where α is a non-zero 1-form. Such an n-dimensional Riemannian manifold is called a special weakly Ricci symmetric manifold and is denoted by $(SWRS)n$.

Let a Riemannian manifold be projectively flat, then

$$P(Y,Z,V) = 0. \tag{3.2}$$

By virtue of (3.2) the relation (1.11) reduces to

$$K(Y,Z,V) = \frac{1}{n-1}\big[\operatorname{Ric}(Z,V)Y - \operatorname{Ric}(Y,V)Z\big]. \tag{3.3}$$

Taking covariant derivative of (3.3) with respect to X, we have

$$(D_X K)(Y,Z,V) = \frac{1}{n-1}\big[(D_X \operatorname{Ric})(Z,V,)Y - (D_X \operatorname{Ric})(Y,V)Z\big]. \tag{3.4}$$

Permuting twice the vectors X, Y, Z; in equation (3.4), then adding the three obtained equations and using Bianchi's second identity, we have

$$\begin{aligned}(D_X \operatorname{Ric})(Z,V)Y + (D_Y \operatorname{Ric})(X,V)Z + (D_Z \operatorname{Ric})(Y,V)X \\ -(D_X \operatorname{Ric})(Y,V)Z - (D_Y \operatorname{Ric})(Z,V)X - (D_Z \operatorname{Ric})(X,Y)X = 0.\end{aligned} \tag{3.5}$$

Using (3.1), in (3.5), we have

$$\begin{aligned}\alpha(X)\operatorname{Ric}(Z,V)Y + \alpha(Y)\operatorname{Ric}(X,V)Z + \alpha(Z)\operatorname{Ric}(Y,V)X \\ -\alpha(X)\operatorname{Ric}(Y,V)Z - \alpha(Y)\operatorname{Ric}(Z,V)X - \alpha(Z)\operatorname{Ric}(X,V)Y = 0.\end{aligned} \tag{3.6}$$

Contracting (3.6) with respect to X, we have

$$\alpha(Z)\operatorname{Ric}(Y,V)-\alpha(Y)\operatorname{Ric}(Z,V)=0. \tag{3.7}$$

Factoring off V in (3.7), we get

$$\alpha(Z)R(Y)-\alpha(Y)R(Z)=0. \tag{3.8}$$

Contracting (3.8) with respect to Y, we have

$$\alpha(Z)r-\alpha(R(Z))=0. \tag{3.9}$$

By virtue of (2.2), the relation (3.9) reduces to

$$g(Z,P)r=g(R(Z),P).$$

Consequently the above equation gives

$$Zr=R(Z).$$

This leads us to the following

Theorem 5. *If the scalar curvature r is constant in a projectively flat $(SWRS)n$ Riemannian manifold, then the Ricci tensor must vanish.*

For a special weakly Ricci symmetric Riemannian manifold, we have (3.1) and if it is an Einstein manifold, then $(D_X \operatorname{Ric})(Y,Z)=0$. Putting $Y=Z=e_i$, in the right hand side of (3.1) and performing a summation over i. We obtain

$$0=\sum_{i=1}^{n}\left\{2\alpha(X)\operatorname{Ric}(e_i,e_i)+\alpha((e_i)\operatorname{Ric}(X,e_i)+\alpha(e_i)\operatorname{Ric}(e_i,X)\right\}$$

or,

$$0=2nc\alpha(X)+\alpha(e_i)c\langle X,e_i\rangle+\alpha(e_i)c\langle_i,X\rangle$$

which reduces to

$$c\big[2n\alpha(X)+\alpha(X)+\alpha(X)\big]=0.$$

But $c\neq 0$, so we have $\alpha(X)=0$, for all X.

Thus, we have the following

Theorem 6. *A special weakly Ricci symmetric Riemannian manifold M cannot be an Einstein manifold if the 1-form $\alpha \neq 0$.*

Taking cyclic of (3.1) and using the symmetry property of Ricci tensor, we have

$$\begin{aligned}(3.10)\quad &(D_X \operatorname{Ric})(Y,Z) + (D_Y \operatorname{Ric})(Z,X) + (D_Z \operatorname{Ric})(X,Y)\\ &\quad = \alpha(X)\operatorname{Ric}(Y,Z) + \alpha(Y)\operatorname{Ric}(Z,X) + \alpha(Z)\operatorname{Ric}(X,Y).\end{aligned}$$

Let $(SWRS)n$ admit a cyclic Ricci tensor, then from (3.10), we have

$$(3.11)\qquad \alpha(X)\operatorname{Ric}(Y,Z) + \alpha(Y)\operatorname{Ric}(Z,X) + \alpha(Z)\operatorname{Ric}(X,Y) = 0.$$

Taking $Y = Z = e_i$ in (3.11) and performing a summation over i, we get

$$\sum_{i=1}^{n}\big[\alpha(X)\operatorname{Ric}(e_i,e_i) + \alpha(e_i)\operatorname{Ric}(e_i X) + \alpha(e_i)\operatorname{Ric}(X,e_i)\big] = 0$$

or

$$nc\alpha(X) + \alpha(e_i)c\langle e_i, X\rangle + \alpha(e_i)c\langle X, e_i\rangle = 0$$

or,

$$c\big[n\alpha(X) + \alpha(X) + \alpha(X)\big] = 0.$$

By virtue of $c \neq 0$, the above equation reduces to

$$(n+2)\alpha(X) = 0.$$

Thus, we have the following

Theorem 7. *If a $(SWRS)n$ admits a cyclic Ricci tensor, then the 1-form α must vanish.*

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